

Entropy numbers of embedding operators of weighted Sobolev spaces with weights that are functions of distance from some h -set

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1 Introduction

In this paper we obtain order estimates for entropy numbers of embedding operator of weighted Sobolev spaces on a John domain into weighted Lebesgue space. Estimates for n -widths of such embeddings were recently obtained in [42, 45].

Definition 1. *Let X, Y be normed spaces, and let $T : X \rightarrow Y$ be a linear continuous operator. Entropy numbers of T are defined by*

$$e_k(T) = \inf \left\{ \varepsilon > 0 : \exists y_1, \dots, y_{2^{k-1}} \in Y : T(B_X) \subset \bigcup_{i=1}^{2^{k-1}} (y_i + \varepsilon B_Y) \right\}, \quad k \in \mathbb{N}.$$

For properties of entropy numbers, we refer the reader to the books [4, 7, 32]. Kolmogorov, Tikhomirov, Birman and Solomyak [2, 15, 36] studied properties of ε -entropy (this magnitude is related to entropy numbers of embedding operators).

Estimates for entropy numbers of the embedding operator of l_p^m into l_q^m were obtained in the paper of Schütt [35] (see also [7]). Here l_p^m ($1 \leq p \leq \infty$) is the space \mathbb{R}^m with the norm

$$\|(x_1, \dots, x_m)\|_q \equiv \|(x_1, \dots, x_m)\|_{l_p^m} = \begin{cases} (|x_1|^p + \dots + |x_m|^p)^{1/p}, & \text{if } p < \infty, \\ \max\{|x_1|, \dots, |x_m|\}, & \text{if } p = \infty. \end{cases}$$

Later Edmunds and Netrusov [5], [6] generalized this result for vector-valued sequence spaces (in particular, for sequence spaces with mixed norm). Haroske, Triebel, Kühn, Leopold, Sickel, Skrzypczak [8–12, 14, 16–23] studied the problem of estimating entropy numbers of embeddings of weighted sequence spaces or weighted Besov and Triebel–Lizorkin spaces.

Lifshits and Linde [25] obtained estimates for entropy numbers of two-weighted Hardy-type operators on a semiaxis (under some conditions on weights). The similar problem for one-weighted Riemann–Liouville operators was considered in the paper of Lomakina and Stepanov [29]. In addition, Lifshits and Linde [26–28] studied the problem of estimating entropy numbers of two-weighted summation operators on a tree.

Triebel [37] and Mieth [31] studied the problem of estimating entropy numbers of embedding operators of weighted Sobolev spaces on a ball with weights that have singularity at the origin.

Estimates of entropy numbers of weighted function spaces are applied in spectral theory of some degenerate elliptic operators (see, e.g., [8, 9, 13, 14, 19, 20, 22]) and in estimating the probability of small deviation of Gaussian random functions (see, e.g., [24, 25, 27]).

The paper is organized as follows. In this section we introduce notations and some basic definitions, and we conclude this section with main result about estimates for entropy numbers of embeddings of weighted Sobolev spaces. In §2 we formulate some known results which will be required in the sequel. In §3 we obtain upper estimates for entropy numbers of embedding operators of some function spaces on a set with tree-like structure (the similar results for n -widths are obtained in [42]). In §4 we prove Theorems 1, 2 and 3 about estimates for entropy numbers of embedding operators of weighted Sobolev spaces. In §5 we obtain estimates of entropy numbers of two-weighted summation operators on a tree.

Let us give the definition of a John domain.

We denote by $AC[t_0, t_1]$ the space of absolutely continuous functions on an interval $[t_0, t_1]$.

Let $B_a(x)$ be the closed euclidean ball of radius a in \mathbb{R}^d centered at the point x .

Definition 2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $a > 0$. We say that $\Omega \in \mathbf{FC}(a)$ if there exists a point $x_* \in \Omega$ such that for any $x \in \Omega$ there exist $T(x) > 0$ and a curve $\gamma_x : [0, T(x)] \rightarrow \Omega$ with the following properties:

1. $\gamma_x \in AC[0, T(x)]$, $\left| \frac{d\gamma_x(t)}{dt} \right| = 1$ a.e.,
2. $\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$,
3. $B_{at}(\gamma_x(t)) \subset \Omega$ for any $t \in [0, T(x)]$.

Definition 3. We say that Ω satisfies the John condition (and call Ω a John domain) if $\Omega \in \mathbf{FC}(a)$ for some $a > 0$.

For a bounded domain the John condition is equivalent to the flexible cone condition (see the definition in [1]). As examples of such domains we can take

1. domains with Lipschitz boundary;
2. the interior of the Koch snowflake;
3. domains $\Omega = \cup_{0 \leq t \leq T} B_{ct}(\gamma(t))$, where $\gamma : [0, T] \rightarrow \mathbb{R}^d$ is a curve with natural parametrization and $c > 0$.

Domains with zero inner angles do not satisfy the John condition.

Reshetnyak [33,34] found the integral representation for smooth functions defined on a John domain Ω in terms of their derivatives of order r . It follows from this integral representation that, for $p > 1$, $1 \leq q < \infty$ and $\frac{r}{d} + \frac{1}{q} - \frac{1}{p} \geq 0$ ($\frac{r}{d} + \frac{1}{q} - \frac{1}{p} > 0$, respectively) the class $W_p^r(\Omega)$ is continuously (respectively, compactly) embedded in the space $L_q(\Omega)$ (i.e., the conditions of the continuous and compact embedding are the same as for $\Omega = [0, 1]^d$).

Introduce the notion of h -set according to [3].

Denote by \mathbb{H} the set of all nondecreasing positive functions defined on $(0, 1]$.

Definition 4. Let $\Gamma \subset \mathbb{R}^d$ be a non-empty compact set, and let $h \in \mathbb{H}$. We say that Γ is an h -set if there are a constant $c_* \geq 1$ and a finite countably additive measure μ on \mathbb{R}^d such that $\text{supp } \mu = \Gamma$ and for any $x \in \Gamma$, $t \in (0, 1]$

$$c_*^{-1}h(t) \leq \mu(B_t(x)) \leq c_*h(t). \quad (1)$$

Example 1. Let $\Gamma \subset \mathbb{R}^d$ be a Lipschitz manifold of dimension k , $0 \leq k < d$. Then Γ is an h -set with $h(t) = t^k$.

Example 2. Let $\Gamma \subset \mathbb{R}^2$ be the Koch snowflake. Then Γ is an h -set with $h(t) = t^{\log 4 / \log 3}$ (see [30, p. 66–68]).

Let us formulate the main result of this paper.

Everywhere below, we use the notation $\log x = \log_2 x$.

Let $|\cdot|$ be a norm on \mathbb{R}^d , and let $E, E' \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$. We set

$$\text{diam}_{|\cdot|} E = \sup\{|y - z| : y, z \in E\}, \quad \text{dist}_{|\cdot|}(x, E) = \inf\{|x - y| : y \in E\}.$$

Let $\Omega \in \mathbf{FC}(a)$ be a bounded domain, and let $\Gamma \subset \partial\Omega$ be an h -set. Further we suppose that in some neighborhood of zero the function $h \in \mathbb{H}$ is defined by

$$h(t) = t^\theta |\log t|^\gamma \tau(|\log t|), \quad 0 \leq \theta < d, \quad (2)$$

where $\tau : (0, +\infty) \rightarrow (0, +\infty)$ is an absolutely continuous function such that

$$\frac{t\tau'(t)}{\tau(t)} \xrightarrow{t \rightarrow +\infty} 0. \quad (3)$$

Let $1 < p \leq \infty$, $1 \leq q < \infty$, $r \in \mathbb{N}$, $\delta := r + \frac{d}{q} - \frac{d}{p} > 0$, $\beta_g, \beta_v \in \mathbb{R}$, $g(x) = \varphi_g(\text{dist}_{|\cdot|}(x, \Gamma))$, $v(x) = \varphi_v(\text{dist}_{|\cdot|}(x, \Gamma))$,

$$\varphi_g(t) = t^{-\beta_g} |\log t|^{-\alpha_g} \rho_g(|\log t|), \quad \varphi_v(t) = t^{-\beta_v} |\log t|^{-\alpha_v} \rho_v(|\log t|), \quad (4)$$

where ρ_g and ρ_v are absolutely continuous functions,

$$\frac{t\rho_g'(t)}{\rho_g(t)} \xrightarrow{t \rightarrow +\infty} 0, \quad \frac{t\rho_v'(t)}{\rho_v(t)} \xrightarrow{t \rightarrow +\infty} 0. \quad (5)$$

In addition, we suppose that

$$\beta_v < \frac{d-\theta}{q} \quad \text{or} \quad \beta_v = \frac{d-\theta}{q}, \quad \alpha_v > \frac{1-\gamma}{q}. \quad (6)$$

Without loss of generality we may assume that $\overline{\Omega} \subset (-\frac{1}{2}, \frac{1}{2})^d$.

We set $\beta = \beta_g + \beta_v$, $\alpha = \alpha_g + \alpha_v$, $\rho(y) = \rho_g(y)\rho_v(y)$, $\mathfrak{Z} = (r, d, p, q, g, v, h, a, c_*)$, $\mathfrak{Z}_* = (\mathfrak{Z}, R)$, where c_* is the constant from Definition 4 and $R = \text{diam } \Omega$.

We use the following notations for order inequalities. Let X, Y be sets, and let $f_1, f_2 : X \times Y \rightarrow \mathbb{R}_+$. We write $f_1(x, y) \underset{y}{\lesssim} f_2(x, y)$ (or $f_2(x, y) \underset{y}{\gtrsim} f_1(x, y)$) if for any $y \in Y$ there exists $c(y) > 0$ such that $f_1(x, y) \leq c(y)f_2(x, y)$ for any $x \in X$; $f_1(x, y) \underset{y}{\asymp} f_2(x, y)$ if $f_1(x, y) \underset{y}{\lesssim} f_2(x, y)$ and $f_2(x, y) \underset{y}{\lesssim} f_1(x, y)$.

Denote by $\mathcal{P}_{r-1}(\mathbb{R}^d)$ the space of polynomials on \mathbb{R}^d of degree not exceeding $r-1$. For a measurable set $E \subset \mathbb{R}^d$ we set

$$\mathcal{P}_{r-1}(E) = \{f|_E : f \in \mathcal{P}_{r-1}(\mathbb{R}^d)\}.$$

Notice that $W_{p,g}^r(\Omega) \supset \mathcal{P}_{r-1}(\Omega)$.

In Theorems 1, 2, 3 the conditions on weights are such that $W_{p,g}^r(\Omega) \subset L_{q,v}(\Omega)$ and there exist $M > 0$ and a linear continuous operator $P : L_{q,v}(\Omega) \rightarrow \mathcal{P}_{r-1}(\Omega)$ such that for any function $f \in W_{p,g}^r(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega)} \leq M \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)} \quad (7)$$

(see [39, 40, 44, 45]).

Remark 1. Let x_* be the point from Definition 2, and let $R' = \text{dist}_{|\cdot|}(x_*, \partial\Omega)$. The operator P defined in [40] (see also [41]) has the following property: there exists $s_0 = s_0(\mathfrak{Z}) \in (0, 1)$ such that, for any function $f \in C^\infty(\Omega) \cap L_{q,v}(\Omega)$ satisfying the condition $f|_{B_{s_0 R'}(x_*)} = 0$, the equality $Pf = 0$ holds.

We set $\hat{W}_{p,g}^r(\Omega) = \{f - Pf : f \in W_{p,g}^r(\Omega)\}$. Let $\hat{\mathcal{W}}_{p,g}^r(\Omega) = \text{span } \hat{W}_{p,g}^r(\Omega)$ be equipped with norm $\|f\|_{\hat{\mathcal{W}}_{p,g}^r(\Omega)} := \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)}$. Denote by $I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)$ the embedding operator. From (7) it follows that I is continuous.

Theorem 1. Let (2), (3), (4), (5), (6) hold and $0 < \theta < d$.

1. Suppose that $\beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$. We set $\alpha_0 = \alpha$ for $\beta_v < \frac{d-\theta}{q}$ and $\alpha_0 = \alpha - \frac{1}{q}$ for $\beta_v = \frac{d-\theta}{q}$. We also suppose that $\frac{\delta}{d} \neq \frac{\delta-\beta}{\theta}$. Denote $\sigma_*(n) = 1$ for $\frac{\delta}{d} < \frac{\delta-\beta}{\theta}$ and

$$\sigma_*(n) = (\log n)^{-\alpha_0 + \frac{(\beta-\delta)\gamma}{\theta}} \rho(\log n) \tau^{\frac{\beta-\delta}{\theta}}(\log n)$$

for $\frac{\delta}{d} > \frac{\delta-\beta}{\theta}$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\min\{\frac{\delta}{d}, \frac{\delta-\beta}{\theta}\} + \frac{1}{q} - \frac{1}{p}} \sigma_*(n).$$

2. Suppose that $\beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$.

(a) Let $p \geq q$ and $\alpha_0 := \alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right) > 0$ for $\beta_v < \frac{d-\theta}{q}$, $\alpha_0 := \alpha - 1 - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right) > 0$ for $\beta_v = \frac{d-\theta}{q}$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} (\log n)^{-\alpha_0} \rho(\log n) \tau^{-\frac{1}{q} + \frac{1}{p}}(\log n).$$

(b) Let $p < q$ and $\alpha_0 := \alpha > 0$ for $\beta_v < \frac{d-\theta}{q}$, $\alpha_0 := \alpha - \frac{1}{q} > 0$ for $\beta_v = \frac{d-\theta}{q}$. Suppose that $\alpha_0 \neq \frac{1}{p} - \frac{1}{q}$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\alpha_0 - \frac{1}{q} + \frac{1}{p}} \rho(\log n)$$

for $\alpha_0 > \frac{1}{p} - \frac{1}{q}$,

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\alpha_0} \rho(n)$$

for $\alpha_0 < \frac{1}{p} - \frac{1}{q}$.

Now we consider the case $\theta = 0$.

Theorem 2. Let (2), (3), (4), (5) hold and $\theta = 0$, $\beta - \delta < 0$, $\beta_v < \frac{d}{q}$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\frac{r}{d}}.$$

In the case $\theta = 0$, $\beta - \delta = 0$ we suppose that $\rho_g(t) = |\log t|^{-\lambda_g}$, $\rho_v = |\log t|^{-\lambda_v}$, $\tau(t) = |\log t|^\nu$ (in the general case the estimates in assertion 1 of Theorem 3 can be obtained similarly). Denote $\lambda = \lambda_g + \lambda_v$.

Theorem 3. Suppose that (2), (3), (4), (5) hold and $\theta = 0$, $\beta - \delta = 0$, $\beta_v < \frac{d}{q}$.

1. Let $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ > 0$. Suppose that $\frac{\alpha}{1-\gamma} \neq \frac{\delta}{d}$. We set $\sigma_*(n) = 1$ for $\frac{\delta}{d} < \frac{\alpha}{1-\gamma}$ and $\sigma_*(n) = (\log n)^{-\lambda - \frac{\alpha\nu}{1-\gamma}}$ for $\frac{\delta}{d} > \frac{\alpha}{1-\gamma}$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\min\{\frac{\delta}{d}, \frac{\alpha}{1-\gamma}\} + \frac{1}{q} - \frac{1}{p}} \sigma_*(n).$$

2. Suppose that $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ = 0$, $\lambda > (1 - \nu) \left(\frac{1}{q} - \frac{1}{p} \right)_+$.

(a) Let $p \geq q$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} (\log n)^{-\lambda + (1-\nu) \left(\frac{1}{q} - \frac{1}{p} \right)}.$$

(b) Let $p < q$, $\lambda \neq \frac{1}{p} - \frac{1}{q}$. Then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{J}_*}{\asymp} n^{\frac{1}{q}-\frac{1}{p}} (\log n)^{-\lambda+\frac{1}{p}-\frac{1}{q}}$$

for $\lambda > \frac{1}{p} - \frac{1}{q}$,

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{\mathfrak{J}_*}{\asymp} n^{-\lambda}$$

for $\lambda < \frac{1}{p} - \frac{1}{q}$.

If Γ is a singleton, then estimates of entropy numbers are given by formulas from Theorem 3 with $\gamma = 0$, $\tau \equiv 1$. The proof is the same as for Theorem 3. These estimates are the generalization of the result of Triebel [37] (in [37] the case $p = q$ was considered).

Without loss of generality we may assume that $|(x_1, \dots, x_d)| = \max_{1 \leq i \leq d} |x_i|$. Further we shall denote $\text{dist} := \text{dist}_{|\cdot|}$, $\text{diam} := \text{diam}_{|\cdot|}$.

2 Preliminaries

The following properties of entropy numbers are well-known (see, e.g., [7]):

1. if $T : X \rightarrow Y$, $S : Y \rightarrow Z$ are linear continuous operators, then $e_{k+l-1}(ST) \leq e_k(S)e_l(T)$;
2. if $T, S : X \rightarrow Y$ are linear continuous operators, then

$$e_{k+l-1}(S + T) \leq e_k(S) + e_l(T). \quad (8)$$

From property 1 it follows that

$$e_k(ST) \leq \|S\|e_k(T), \quad e_k(ST) \leq \|T\|e_k(S). \quad (9)$$

Further we denote by I_ν the identity operator on \mathbb{R}^ν .

Theorem A. (see [7, 35]). Let $1 \leq p \leq q \leq \infty$. Then

$$e_k(I_\nu : l_p^\nu \rightarrow l_q^\nu) \underset{p,q}{\asymp} \begin{cases} 1, & 1 \leq k \leq \log \nu, \\ \left(\frac{\log(1+\frac{\nu}{k})}{k} \right)^{\frac{1}{p}-\frac{1}{q}}, & \log \nu \leq k \leq \nu, \\ 2^{-\frac{k}{\nu}} \nu^{\frac{1}{q}-\frac{1}{p}}, & \nu \leq k. \end{cases}$$

Let $1 \leq q < p \leq \infty$. Then

$$e_k(I_\nu : l_p^\nu \rightarrow l_q^\nu) \underset{p,q}{\asymp} 2^{-\frac{k}{\nu}} \nu^{\frac{1}{q}-\frac{1}{p}}, \quad k \in \mathbb{N}.$$

Remark 2. In estimates from [7] the value 2ν was taken instead of ν since the spaces l_p^ν, l_q^ν were considered as spaces over \mathbb{C} .

In the paper of Kühn [23] the order estimates for entropy numbers of diagonal operators $D_\sigma : l_p \rightarrow l_q$ were obtained for $p > q$.

Theorem B. (see [23]). Let $0 < q < p \leq \infty, \sigma = (\sigma_k)_{k \in \mathbb{N}} \in l_{\frac{pq}{p-q}}, \omega_n = \left(\sum_{k=n}^{\infty} \sigma_k^{\frac{pq}{p-q}} \right)^{\frac{1}{q} - \frac{1}{p}}$. Suppose that there exists $C > 0$ such that $\omega_n \leq C\omega_{2n}$ for any n . We define the operator $D_\sigma : l_p \rightarrow l_q$ by $D_\sigma(x_k)_{k \in \mathbb{N}} = (\sigma_k x_k)_{k \in \mathbb{N}}$. Then $e_n(D_\sigma : l_p \rightarrow l_q) \underset{C,p,q}{\asymp} \omega_n$.

The following result was proved by Lifshits [26].

Theorem C. (see [26]). Let X, Y be normed spaces, and let $V \in L(X, Y), \{V_\nu\}_{\nu \in \mathcal{N}} \subset L(X, Y)$. Then for any $n \in \mathbb{N}$

$$e_{n+[\log_2 |\mathcal{N}|]+1}(V) \leq \sup_{\nu \in \mathcal{N}} e_n(V_\nu) + \sup_{x \in B_X} \inf_{\nu \in \mathcal{N}} \|Vx - V_\nu x\|_Y.$$

3 Estimates for entropy numbers of function classes on a set with tree-like structure

First we give some notations.

Let $(\Omega, \Sigma, \text{mes})$ be a measure space. We say that sets $A, B \subset \Omega$ are disjoint if $\text{mes}(A \cap B) = 0$. Let $E, E_1, \dots, E_m \subset \Omega$ be measurable sets, and let $m \in \mathbb{N} \cup \{\infty\}$. We say that $\{E_i\}_{i=1}^m$ is a partition of E if the sets E_i are pairwise disjoint and $\text{mes}((\cup_{i=1}^m E_i) \triangle E) = 0$.

Denote by $\chi_E(\cdot)$ the indicator function of a set E .

Let \mathcal{G} be a graph containing at most countable number of vertices. We shall denote by $\mathbf{V}(\mathcal{G})$ and by $\mathbf{E}(\mathcal{G})$ the set of vertices and the set of edges of \mathcal{G} , respectively. Two vertices are called *adjacent* if there is an edge between them. Let $\xi_i \in \mathbf{V}(\mathcal{G}), 1 \leq i \leq n$. The sequence (ξ_1, \dots, ξ_n) is called a *path* if the vertices ξ_i and ξ_{i+1} are adjacent for any $i = 1, \dots, n-1$. If all the vertices ξ_i are distinct, then such a path is called *simple*.

Let (\mathcal{T}, ξ_0) be a tree with a distinguished vertex (or a root) ξ_0 . We introduce a partial order on $\mathbf{V}(\mathcal{T})$ as follows: we say that $\xi' > \xi$ if there exists a simple path $(\xi_0, \xi_1, \dots, \xi_n, \xi')$ such that $\xi = \xi_k$ for some $k \in \overline{0, n}$. In this case, we set $\rho_{\mathcal{T}}(\xi, \xi') = \rho_{\mathcal{T}}(\xi', \xi) = n + 1 - k$. In addition, we denote $\rho_{\mathcal{T}}(\xi, \xi) = 0$. If $\xi' > \xi$ or $\xi' = \xi$, then we write $\xi' \geq \xi$. This partial order on \mathcal{T} induces a partial order on its subtree.

Given $j \in \mathbb{Z}_+, \xi \in \mathbf{V}(\mathcal{T})$, we denote

$$\mathbf{V}_j(\xi) := \mathbf{V}_j^{\mathcal{T}}(\xi) := \{\xi' \geq \xi : \rho_{\mathcal{T}}(\xi, \xi') = j\}.$$

For $\xi \in \mathbf{V}(\mathcal{T})$ we denote by $\mathcal{T}_\xi = (\mathcal{T}_\xi, \xi)$ the subtree in \mathcal{T} with vertex set

$$\{\xi' \in \mathbf{V}(\mathcal{T}) : \xi' \geq \xi\}. \quad (10)$$

Let \mathcal{G} be a subgraph in \mathcal{T} . Denote by $\mathbf{V}_{\max}(\mathcal{G})$ and $\mathbf{V}_{\min}(\mathcal{G})$ the sets of maximal and minimal vertices in \mathcal{G} , respectively.

Let $\mathbf{W} \subset \mathbf{V}(\mathcal{T})$. We say that $\mathcal{G} \subset \mathcal{T}$ is a maximal subgraph on the set of vertices \mathbf{W} if $\mathbf{V}(\mathcal{G}) = \mathbf{W}$ and any two vertices $\xi', \xi'' \in \mathbf{W}$ adjacent in \mathcal{T} are also adjacent in \mathcal{G} .

Let $\{\mathcal{T}_j\}_{j \in \mathbb{N}}$ be a family of subtrees in \mathcal{T} such that $\mathbf{V}(\mathcal{T}_j) \cap \mathbf{V}(\mathcal{T}_{j'}) = \emptyset$ for $j \neq j'$ and $\cup_{j \in \mathbb{N}} \mathbf{V}(\mathcal{T}_j) = \mathbf{V}(\mathcal{T})$. Then $\{\mathcal{T}_j\}_{j \in \mathbb{N}}$ is called a partition of the tree \mathcal{T} . Let ξ_j be the minimal vertex of \mathcal{T}_j . We say that the tree \mathcal{T}_s succeeds the tree \mathcal{T}_j (or \mathcal{T}_j precedes the tree \mathcal{T}_s) if $\xi_j < \xi_s$ and

$$\{\xi \in \mathcal{T} : \xi_j \leq \xi < \xi_s\} \subset \mathbf{V}(\mathcal{T}_j).$$

We consider the function spaces on sets with tree-like structure from [42].

Let $(\Omega, \Sigma, \text{mes})$ be a measure space, let $\hat{\Theta}$ be a countable partition of Ω into measurable subsets, let \mathcal{A} be a tree with a root such that

$$\exists c_1 \geq 1 : \quad \text{card } \mathbf{V}_1(\xi) \leq c_1, \quad \xi \in \mathbf{V}(\mathcal{A}), \quad (11)$$

and let $\hat{F} : \mathbf{V}(\mathcal{A}) \rightarrow \hat{\Theta}$ be a bijective mapping.

Throughout we consider at most countable partitions into measurable subsets.

Let $1 < p \leq \infty$, $1 \leq q < \infty$ be arbitrary numbers. We suppose that, for any measurable subset $E \subset \Omega$, the following spaces are defined:

- the space $X_p(E)$ with seminorm $\|\cdot\|_{X_p(E)}$,
- the space $Y_q(E)$ with seminorm $\|\cdot\|_{Y_q(E)}$,

which all satisfy the following conditions:

1. $X_p(\Omega) \subset Y_q(\Omega)$;
2. $X_p(E) = \{f|_E : f \in X_p(\Omega)\}$, $Y_q(E) = \{f|_E : f \in Y_q(\Omega)\}$;
3. if $\text{mes } E = 0$, then $\dim Y_q(E) = \dim X_p(E) = 0$;
4. if $E \subset \Omega$, $E_j \subset \Omega$ ($j \in \mathbb{N}$) are measurable subsets, $E = \sqcup_{j \in \mathbb{N}} E_j$, then

$$\|f\|_{X_p(E)} = \left\| \left\{ \|f|_{E_j}\|_{X_p(E_j)} \right\}_{j \in \mathbb{N}} \right\|_{l_p}, \quad f \in X_p(E), \quad (12)$$

$$\|f\|_{Y_q(E)} = \left\| \left\{ \|f|_{E_j}\|_{Y_q(E_j)} \right\}_{j \in \mathbb{N}} \right\|_{l_q}, \quad f \in Y_q(E); \quad (13)$$

5. if $E \in \Sigma$, $f \in Y_q(\Omega)$, then $f \cdot \chi_E \in Y_q(\Omega)$.

Let $\mathcal{P}(\Omega) \subset X_p(\Omega)$ be a subspace of finite dimension r_0 and let $\|f\|_{X_p(\Omega)} = 0$ for any $f \in \mathcal{P}(\Omega)$. For each measurable subset $E \subset \Omega$ we write $\mathcal{P}(E) = \{P|_E : P \in \mathcal{P}(\Omega)\}$. Let $G \subset \Omega$ be a measurable subset and let T be a partition of G . We set

$$\mathcal{S}_T(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f|_E \in \mathcal{P}(E), f|_{\Omega \setminus G} = 0\}. \quad (14)$$

If T is finite, then $\mathcal{S}_T(\Omega) \subset Y_q(\Omega)$ (see property 5).

For any finite partition $T = \{E_j\}_{j=1}^n$ of the set E and for each function $f \in Y_q(\Omega)$ we put

$$\|f\|_{p,q,T} = \left(\sum_{j=1}^n \|f|_{E_j}\|_{Y_q(E_j)}^{\sigma_{p,q}} \right)^{\frac{1}{\sigma_{p,q}}},$$

where $\sigma_{p,q} = \min\{p, q\}$. Denote by $Y_{p,q,T}(E)$ the space $Y_q(E)$ with the norm $\|\cdot\|_{p,q,T}$. Notice that $\|\cdot\|_{Y_q(E)} \leq \|\cdot\|_{p,q,T}$.

For each subtree $\mathcal{A}' \subset \mathcal{A}$ we set $\Omega_{\mathcal{A}'} = \cup_{\xi \in \mathbf{V}(\mathcal{A}')} \hat{F}(\xi)$.

Assumption 1. *There is a function $w_* : \mathbf{V}(\mathcal{A}) \rightarrow (0, \infty)$ with the following property: for any $\hat{\xi} \in \mathbf{V}(\mathcal{A})$ there exists a linear continuous operator $P_{\Omega_{\mathcal{A}_{\hat{\xi}}}} : Y_q(\Omega) \rightarrow \mathcal{P}(\Omega)$ such that for any function $f \in X_p(\Omega)$ and any subtree $\mathcal{A}' \subset \mathcal{A}$ rooted at $\hat{\xi}$*

$$\|f - P_{\Omega_{\mathcal{A}_{\hat{\xi}}}} f\|_{Y_q(\Omega_{\mathcal{A}'})} \leq w_*(\hat{\xi}) \|f\|_{X_p(\Omega_{\mathcal{A}'})}. \quad (15)$$

Assumption 2. *There exist a function $\tilde{w}_* : \mathbf{V}(\mathcal{A}) \rightarrow (0, \infty)$ and numbers $\delta_* > 0$, $c_2 \geq 1$ such that for each vertex $\xi \in \mathbf{V}(\mathcal{A})$ and for any $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$ there is a partition $T_{m,n}(G)$ of the set $G = \hat{F}(\xi)$ with the following properties:*

1. $\text{card } T_{m,n}(G) \leq c_2 \cdot 2^m n$.

2. *For any $E \in T_{m,n}(G)$ there exists a linear continuous operator $P_E : Y_q(\Omega) \rightarrow \mathcal{P}(E)$ such that for any function $f \in X_p(\Omega)$*

$$\|f - P_E f\|_{Y_q(E)} \leq (2^m n)^{-\delta_*} \tilde{w}_*(\xi) \|f\|_{X_p(E)}. \quad (16)$$

3. *For any $E \in T_{m,n}(G)$*

$$\text{card } \{E' \in T_{m\pm 1,n}(G) : \text{mes}(E \cap E') > 0\} \leq c_2. \quad (17)$$

Assumption 3. *There exist $k_* \in \mathbb{N}$, $\lambda_* \geq 0$,*

$$\mu_* \geq \lambda_*, \quad (18)$$

$\gamma_* > 0$, absolutely continuous functions $u_* : (0, \infty) \rightarrow (0, \infty)$ and $\psi_* : (0, \infty) \rightarrow (0, \infty)$, $c_3 \geq 1$, $t_0 \in \mathbb{N}$, a partition $\{\mathcal{A}_{t,i}\}_{t \geq t_0, i \in \hat{J}_t}$ of the tree \mathcal{A} such that $\lim_{y \rightarrow \infty} \frac{yu'_*(y)}{u_*(y)} = 0$, $\lim_{y \rightarrow \infty} \frac{y\psi'_*(y)}{\psi_*(y)} = 0$,

$$c_3^{-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \leq w_*(\xi) \leq c_3 \cdot 2^{-\lambda_* k_* t} u_*(2^{k_* t}), \quad \xi \in \mathbf{V}(\mathcal{A}_{t,i}), \quad (19)$$

$$c_3^{-1} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \leq \tilde{w}_*(\xi) \leq c_3 \cdot 2^{-\mu_* k_* t} u_*(2^{k_* t}), \quad \xi \in \mathbf{V}(\mathcal{A}_{t,i}), \quad (20)$$

and for $\nu_t := \sum_{i \in \hat{J}_t} \text{card } \mathbf{V}(\mathcal{A}_{t,i})$ one of the following estimates holds:

$$\nu_t \leq c_3 \cdot 2^{\gamma_* k_* t} \psi_*(2^{k_* t}) =: c_3 \bar{\nu}_t, \quad t \geq t_0, \quad (21)$$

or

$$k_* = 1, \quad \nu_t \leq c_3 \cdot 2^{\gamma_* 2^t} \psi_*(2^{2^t}) =: c_3 \bar{\nu}_t, \quad t \geq t_0. \quad (22)$$

In addition, we assume that the following assertions hold.

1. If $p > q$, then

$$2^{-\lambda_* k_* t} (\text{card } \hat{J}_t)^{\frac{1}{q} - \frac{1}{p}} \leq c_3 \cdot 2^{-\mu_* k_* t} \bar{\nu}_t^{\frac{1}{q} - \frac{1}{p}}. \quad (23)$$

2. Let $t, t' \in \mathbb{Z}_+$. Then

$$2^{-\lambda_* k_* t'} u_*(2^{k_* t'}) \leq c_3 \cdot 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \quad \text{if } t' \geq t, \quad (24)$$

$$\begin{aligned} & 2^{-\mu_* k_* t'} u_*(2^{k_* t'}) \bar{\nu}_{t'}^{\frac{1}{q} - \frac{1}{p}} \leq \\ & \leq c_3 \cdot 2^{-\mu_* k_* t} u_*(2^{k_* t}) \bar{\nu}_t^{\frac{1}{q} - \frac{1}{p}} \quad \text{if } t' \geq t, \quad p > q. \end{aligned} \quad (25)$$

3. If the tree $\mathcal{A}_{t',i'}$ succeeds the tree $\mathcal{A}_{t,i}$, then $t' = t + 1$.

Remark 3. If $\xi \in \mathbf{V}(\mathcal{A}_{t,i})$, $\xi' \in \mathbf{V}(\mathcal{A}_{t',i'})$, $\xi' > \xi$, then $t' > t$.

Remark 4. If $p > q$, then from (25) it follows that (22) cannot hold.

We introduce some more notation.

- $\hat{\xi}_{t,i}$ is the minimal vertex of the tree $\mathcal{A}_{t,i}$.
- Γ_t is the maximal subgraph in \mathcal{A} on the set of vertices $\cup_{i \in \hat{J}_t} \mathbf{V}(\mathcal{A}_{t,i})$, $t \geq t_0$; for $1 \leq t < t_0$ we put $\Gamma_t = \emptyset$, $\hat{J}_t = \emptyset$.
- $G_t = \cup_{\xi \in \mathbf{V}(\Gamma_t)} \hat{F}(\xi) = \cup_{i \in \hat{J}_t} \Omega_{\mathcal{A}_{t,i}}$.

- $\tilde{\Gamma}_t$ is the maximal subgraph on the set of vertices $\cup_{j \geq t} \mathbf{V}(\Gamma_j)$, $t \in \mathbb{N}$.
- $\{\tilde{\mathcal{A}}_{t,i}\}_{i \in \bar{J}_t}$ is the set of connected components of the graph $\tilde{\Gamma}_t$.
- $\tilde{U}_{t,i} = \cup_{\xi \in \mathbf{V}(\tilde{\mathcal{A}}_{t,i})} \hat{F}(\xi)$.
- $\tilde{U}_t = \cup_{i \in \bar{J}_t} \tilde{U}_{t,i} = \cup_{\xi \in \mathbf{V}(\tilde{\Gamma}_t)} \hat{F}(\xi)$.

If $t \geq t_0$, then

$$\mathbf{V}_{\min}(\tilde{\Gamma}_t) = \mathbf{V}_{\min}(\Gamma_t) = \{\hat{\xi}_{t,i}\}_{i \in \hat{J}_t} \quad (26)$$

(see [42, p. 30]), and we may assume that

$$\bar{J}_t = \hat{J}_t, \quad t \geq t_0. \quad (27)$$

The set \hat{J}_{t_0} is a singleton. Denote $\{i_0\} = \hat{J}_{t_0}$.

We set $\mathfrak{Z}_0 = (p, q, r_0, w_*, \tilde{w}_*, \delta_*, k_*, \lambda_*, \mu_*, \gamma_*, \psi_*, u_*, c_1, c_2, c_3)$.

From Assumption 1 it follows that there exists a linear continuous operator $\hat{P} : Y_q(\Omega) \rightarrow \mathcal{P}(\Omega)$ such that for any function $f \in X_p(\Omega)$

$$\|f - \hat{P}f\|_{Y_q(\Omega)} \underset{\mathfrak{Z}_0}{\lesssim} \|f\|_{X_p(\Omega)} \quad (28)$$

(we take as \hat{P} the operator $P_{\Omega, \mathcal{A}_{\xi_{i_0}, i_0}}$). We set

$$\hat{X}_p(\Omega) = \{f - \hat{P}f : f \in X_p(\Omega)\}.$$

Then $\hat{X}_p(\Omega) \subset X_p(\Omega)$. Moreover, since $Y_q(\Omega)$ is a normed space and $\|f\|_{X_p(\Omega)} = \|f - \hat{P}f\|_{X_p(\Omega)}$ (by the property of $\mathcal{P}(\Omega)$), then $(\hat{X}_p(\Omega), \|\cdot\|_{X_p(\Omega)})$ is a normed space.

Indeed, if $\|f - \hat{P}f\|_{X_p(\Omega)} = 0$, then $\|f - \hat{P}f\|_{Y_q(\Omega)} \stackrel{(28)}{=} 0$ and $f - \hat{P}f = 0$.

Denote by $B\hat{X}_p(\Omega)$ the unit ball of $\hat{X}_p(\Omega)$. Let $I : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)$ be the embedding operator.

The following assertions were proved in [42, p. 30, 32].

Lemma 1. *There exists $x_0 \in (0, \infty)$ such that for any $x \geq x_0$ the equation $y^{\gamma_*} \psi_*(y) = x$ has a unique solution $y(x)$. Moreover, $y(x) = x^{\beta_*} \varphi_*(x)$, where $\beta_* = \frac{1}{\gamma_*}$ and φ_* is an absolutely continuous function such that $\lim_{x \rightarrow +\infty} \frac{x \varphi'_*(x)}{\varphi_*(x)} = 0$.*

Lemma 2. *Let $\gamma_* > 0$, $\psi_*(y) = |\log y|^{\alpha_*} \rho_*(|\log y|)$, where $\rho_* : (0, \infty) \rightarrow (0, \infty)$ is an absolutely continuous function such that $\lim_{y \rightarrow \infty} \frac{y \rho'_*(y)}{\rho_*(y)} = 0$. Let φ_* be such as in Lemma 1. Then for sufficiently large $x > 1$*

$$\varphi_*(x) \underset{\gamma_*, \alpha_*, \rho_*}{\asymp} (\log x)^{-\frac{\alpha_*}{\gamma_*}} [\rho_*(\log x)]^{-\frac{1}{\gamma_*}}.$$

Theorem 4. Let $1 < p \leq \infty$, $1 \leq q < \infty$, let assumptions 1, 2 and 3 with (21) hold, and let

$$\delta_* > \left(\frac{1}{q} - \frac{1}{p} \right)_+. \quad (29)$$

Suppose that $\delta_* \neq \lambda_* \beta_*$ for $p \leq q$ and $\delta_* \neq \mu_* \beta_*$ for $p > q$. Then there exists $n_0 = n_0(\mathfrak{Z}_0)$ such that for any $n \geq n_0$ the following estimates hold.

- Let $p \leq q$. We set

$$\sigma_*(n) = \begin{cases} 1 & \text{for } \delta_* < \lambda_* \beta_*, \\ u_*(n^{\beta_*} \varphi_*(n)) \varphi_*^{-\lambda_*}(n) & \text{for } \delta_* > \lambda_* \beta_*. \end{cases}$$

Then

$$e_n(I : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\min(\delta_*, \lambda_* \beta_*) + \frac{1}{q} - \frac{1}{p}} \sigma_*(n). \quad (30)$$

- Let $p > q$. We set

$$\sigma_*(n) = \begin{cases} 1 & \text{for } \delta_* < \mu_* \beta_*, \\ u_*(n^{\beta_*} \varphi_*(n)) \varphi_*^{-\mu_*}(n) & \text{for } \delta_* > \mu_* \beta_*. \end{cases}$$

Then

$$e_n(I : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\min(\delta_*, \mu_* \beta_*) + \frac{1}{q} - \frac{1}{p}} \sigma_*(n). \quad (31)$$

Theorem 5. Let $1 < p < q < \infty$, let assumptions 1, 2 and 3 with (22) hold, and let $\delta_* > \left(\frac{1}{q} - \frac{1}{p} \right)_+$. Then

$$e_n(I : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* - \frac{1}{q} + \frac{1}{p}} u_*(\log n) \quad (32)$$

if $\lambda_* > \frac{1}{p} - \frac{1}{q}$, and

$$e_n(I : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_*} u_*(n) \quad (33)$$

if $\lambda_* < \frac{1}{p} - \frac{1}{q}$.

First we prove some auxiliary assertions.

Definition of operators Q_t . By assumption 1 and (19), for any $t \geq t_0$, $i \in \hat{J}_t \stackrel{(27)}{=} \bar{J}_t$ there exists a linear continuous operator $\tilde{P}_{t,i} : Y_q(\Omega) \rightarrow \mathcal{P}(\Omega)$ such that for any function $f \in \hat{X}_p(\Omega)$ and for any subtree $\mathcal{A}' \subset \mathcal{A}$ rooted at $\hat{\xi}_{t,i}$

$$\|f - \tilde{P}_{t,i} f\|_{Y_q(\Omega_{\mathcal{A}'})} \underset{\mathfrak{Z}_0}{\lesssim} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \|f\|_{X_p(\Omega_{\mathcal{A}'})}; \quad (34)$$

in particular,

$$\|f - \tilde{P}_{t,i}f\|_{Y_q(\tilde{U}_{t,i})} \underset{3_0}{\lesssim} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \|f\|_{X_p(\tilde{U}_{t,i})}. \quad (35)$$

Moreover, from the definitions of the space $\hat{X}_p(\Omega)$ and of the operator \hat{P} it follows that we can set $\tilde{P}_{t_0,i_0} = 0$. Let $1 \leq t < t_0$. Then $\bar{J}_t = \{i_0\}$, $\tilde{U}_{t,i_0} = \tilde{U}_{t_0,i_0}$. We set $\tilde{P}_{t,i_0} = 0$. By (24) we get that (35) holds.

Let

$$Q_t f(x) = \tilde{P}_{t,i} f(x) \quad \text{for } x \in \tilde{U}_{t,i}, \quad i \in \bar{J}_t, \quad Q_t f(x) = 0 \quad \text{for } x \in \Omega \setminus \tilde{U}_t, \quad (36)$$

$$T_t = \{\tilde{U}_{t+1,i}\}_{i \in \bar{J}_{t+1}}. \quad (37)$$

Then $(Q_{t+1}f - Q_t f)\chi_{\tilde{U}_{t+1}} \in \mathcal{S}_{T_t}(\Omega)$, and for $p \leq q$ we have

$$\|f - Q_t f\|_{Y_q(\tilde{U}_t)} \leq \|f - Q_t f\|_{Y_{p,q,T_{t-1}}(\tilde{U}_t)} \overset{(35)}{\underset{3_0}{\lesssim}} 2^{-\lambda_* k_* t} u_*(2^{k_* t}). \quad (38)$$

Notice that if $t < t_0$, then $Q_t f = Q_{t+1} f = 0$ (since $\tilde{P}_{t,i_0} = 0$ for $t \leq t_0$ by definition).

Definition of operators $P_{t,m}$. For $t \geq t_0$ we set

$$m_t = \lceil \log \nu_t \rceil. \quad (39)$$

In [42] for each $m \in \mathbb{Z}_+$ the set $G_{m,t} \subset G_t$, the partition $\tilde{T}_{t,m}$ of $G_{m,t}$ and the linear continuous operator $P_{t,m} : Y_q(\Omega) \rightarrow \mathcal{S}_{\tilde{T}_{t,m}}(\Omega)$ were constructed with the following properties:

1. $G_{m,t} \subset G_{m+1,t}$, $G_{m_t,t} = G_t$;
2. for any $m \in \mathbb{Z}_+$

$$\text{card } \tilde{T}_{t,m} \underset{3_0}{\lesssim} 2^m; \quad (40)$$

3. for any function $f \in \hat{X}_p(\Omega)$ and for any set $E \in \tilde{T}_{t,m}$

$$\|f - P_{t,m} f\|_{Y_q(E)} \underset{3_0}{\lesssim} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \|f\|_{X_p(E)}, \quad m \leq m_t, \quad (41)$$

$$\|f - P_{t,m} f\|_{Y_q(E)} \underset{3_0}{\lesssim} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_*(m-m_t)} \|f\|_{X_p(E)}, \quad m > m_t; \quad (42)$$

4. for any set $E \in \tilde{T}_{t,m}$

$$\text{card } \{E' \in \tilde{T}_{t,m \pm 1} : \text{mes}(E \cap E') > 0\} \underset{3_0}{\lesssim} 1. \quad (43)$$

Moreover, we may assume that $\tilde{T}_{m_i, t} = \{\hat{F}(\xi)\}_{\xi \in \mathbf{V}(\Gamma_t)}$.

Remark 5. In [42] the relation (42) was proved with λ_* instead of μ_* ; however it follows from the construction and from assumption 2 that the estimate with μ_* is also correct (see [42, proof of formula (80)]).

Let

$$t_*(n) = \min\{t \in \mathbb{N} : \bar{\nu}_t \geq n\}, \quad (44)$$

$$t_{**}(n) = \begin{cases} t_*(n), & \text{if } p \geq q, \\ \min\{t \in \mathbb{N} : \bar{\nu}_t \geq 2^n\}, & \text{if } p < q. \end{cases} \quad (45)$$

The following lemma is proved in [43, formula (60)].

Lemma 3. Let $\Lambda_* : (0, +\infty) \rightarrow (0, +\infty)$ be an absolutely continuous function such that $\lim_{y \rightarrow +\infty} \frac{y\Lambda'_*(y)}{\Lambda_*(y)} = 0$. Then for any $\varepsilon > 0$

$$t^{-\varepsilon} \underset{\varepsilon, \Lambda_*}{\lesssim} \frac{\Lambda_*(ty)}{\Lambda_*(y)} \underset{\varepsilon, \Lambda_*}{\lesssim} t^\varepsilon, \quad 1 \leq y < \infty, \quad 1 \leq t < \infty. \quad (46)$$

Proposition 1. If (21) holds, then

$$\bar{\nu}_{t_*(n)} \underset{3}{\asymp} n, \quad 2^{k_* t_*(n)} \underset{3}{\asymp} n^{\beta_*} \varphi_*(n), \quad (47)$$

and in the case $p < q$ we have

$$\bar{\nu}_{t_{**}(n)} \underset{3}{\asymp} 2^n, \quad 2^{k_* t_{**}(n)} \underset{3}{\asymp} 2^{\beta_* n} \varphi_*(2^n). \quad (48)$$

If (22) holds, then for sufficiently large $n \in \mathbb{N}$

$$2^{t_*(n)} \underset{3}{\asymp} \log n; \quad \text{if } p < q, \quad \text{then } 2^{t_{**}(n)} \underset{3}{\asymp} n. \quad (49)$$

Proof. Estimates (47) follow from [42, formula (51)]. The relations (48) can be proved similarly.

Let us prove the first estimate in (49); the second relation is proved similarly. We have

$$n \stackrel{(22), (44)}{\leq} 2^{\gamma_* \cdot 2^{t_*(n)}} \psi_*(2^{2^{t_*(n)}}) \stackrel{(46)}{\underset{3_0}{\lesssim}} 2^{2^{\gamma_* \cdot 2^{t_*(n)}}};$$

therefore, $2^{t_*(n)} \underset{3_0}{\gtrsim} \log n$ for sufficiently large $n \in \mathbb{N}$. Further,

$$n \stackrel{(22), (44)}{>} 2^{\gamma_* \cdot 2^{t_*(n)-1}} \psi_*(2^{2^{t_*(n)-1}}) \stackrel{(46)}{\underset{3_0}{\gtrsim}} 2^{\gamma_* \cdot 2^{t_*(n)-2}},$$

and $2^{t_*(n)} \underset{3_0}{\lesssim} \log n$ for sufficiently large $n \in \mathbb{N}$. □

It is proved in [42, p. 37–39] that for $f \in \hat{X}_p(\Omega)$

$$f = \sum_{j=t_0}^{t_{**}(n)-1} (Q_{j+1}f - Q_jf)\chi_{\tilde{U}_{j+1}} + \sum_{j=t_0}^{t_{**}(n)-1} (f - Q_jf)\chi_{G_j} + (f - Q_{t_{**}(n)}f)\chi_{\tilde{U}_{t_{**}(n)}}, \quad (50)$$

$$\begin{aligned} \sum_{t=t_0}^{t_{**}(n)-1} (f - Q_t f)\chi_{G_t} &= \sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} (P_{t,m+1}f - P_{t,m}f)\chi_{G_{m,t}} + \\ &+ \sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{m=m_t}^{\infty} (P_{t,m+1}f - P_{t,m}f)\chi_{G_t} + \sum_{t=t_*(n)}^{t_{**}(n)-1} (P_{t,m_t}f - Q_t f)\chi_{G_t}, \end{aligned} \quad (51)$$

$$\begin{aligned} \sum_{t=t_0}^{t_{**}(n)-1} (f - Q_t f)\chi_{G_t} &= \sum_{t=t_0}^{t_*(n)-1} \sum_{m=m_t+1}^{\infty} (P_{t,m+1}f - P_{t,m}f)\chi_{G_{m,t}} + \\ &+ \sum_{t=t_0}^{t_*(n)-1} (P_{t,m_t+1}f - Q_t f)\chi_{G_t} + \sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{m=m_t}^{\infty} (P_{t,m+1}f - P_{t,m}f)\chi_{G_t} + \\ &+ \sum_{t=t_*(n)}^{t_{**}(n)-1} (P_{t,m_t}f - Q_t f)\chi_{G_t}. \end{aligned} \quad (52)$$

Lemma 4. 1. Let (21) hold. Then for any $f \in B\hat{X}_p(\Omega)$

$$\|f - Q_{t_{**}(n)}f\|_{Y_q(\tilde{U}_{t_{**}(n)})} \lesssim_{3_0} n^{-\mu_*\beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)) \quad \text{if } p > q, \quad (53)$$

$$\|f - Q_{t_{**}(n)}f\|_{Y_q(\tilde{U}_{t_{**}(n)})} \lesssim_{3_0} n^{-\lambda_*\beta_*} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)) \quad \text{if } p = q, \quad (54)$$

$$\|f - Q_{t_{**}(n)}f\|_{Y_q(\tilde{U}_{t_{**}(n)})} \lesssim_{3_0} 2^{-\lambda_*\beta_*n} \varphi_*^{-\lambda_*}(2^n) u_*(2^{\beta_*n} \varphi_*(2^n)) \quad \text{if } p < q. \quad (55)$$

2. Let (22) hold (by Remark 4, this case is possible only for $p \leq q$). Then for $f \in B\hat{X}_p(\Omega)$

$$\|f - Q_{t_{**}(n)}f\|_{Y_q(\tilde{U}_{t_{**}(n)})} \lesssim_{3_0} (\log n)^{-\lambda_*} u_*(\log n) \quad \text{if } p = q, \quad (56)$$

$$\|f - Q_{t_{**}(n)}f\|_{Y_q(\tilde{U}_{t_{**}(n)})} \lesssim_{3_0} n^{-\lambda_*} u_*(n) \quad \text{if } p < q. \quad (57)$$

Proof. Estimates from assertion 1 follow from (38), (46), (47), (48). Estimates from assertion 2 follow from (38), (46) and (49). \square

Recall the notation $\sigma_{p,q} = \min\{p, q\}$.

Lemma 5. (see [42]). Let T be a finite partition of a measurable subset $G \subset \Omega$, $\nu = \dim \mathcal{S}_T(\Omega)$ (see (14)). Then there exists a linear isomorphism $A : \mathcal{S}_T(\Omega) \rightarrow \mathbb{R}^\nu$ such that $\|A\|_{Y_{p,q,T}(G) \rightarrow l_{\sigma_{p,q}}^\nu} \lesssim 1$, $\|A^{-1}\|_{l_q^\nu \rightarrow Y_q(G)} \lesssim 1$.

Lemma 6. There exists a sequence $\{k_t\}_{t=t_0}^{t_{**}(n)-1} \subset \mathbb{N}$ such that

$$\sum_{t=t_0}^{t_{**}(n)-1} (k_t - 1) \lesssim_{3_0} n \quad (58)$$

and the following assertions hold.

1. Suppose that (21) holds. Then for $p > q$

$$\sum_{t=t_0}^{t_{**}(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(\tilde{U}_{t+1})) \lesssim_{3_0} n^{-\mu_*\beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)), \quad (59)$$

and for $p \leq q$

$$\sum_{t=t_0}^{t_{**}(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(\tilde{U}_{t+1})) \lesssim_{3_0} n^{-\lambda_*\beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)). \quad (60)$$

2. Suppose that (22) holds (by Remark 4, this case is possible only for $p \leq q$). If $p = q$, then

$$\sum_{t=t_0}^{t_{**}(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(\tilde{U}_{t+1})) \lesssim_{3_0} (\log n)^{-\lambda_*} u_*(\log n); \quad (61)$$

if $p < q$ and $\lambda_* > \frac{1}{p} - \frac{1}{q}$, then

$$\sum_{t=t_0}^{t_{**}(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(\tilde{U}_{t+1})) \lesssim_{3_0} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p} - \frac{1}{q}} u_*(\log n); \quad (62)$$

if $p < q$ and $\lambda_* < \frac{1}{p} - \frac{1}{q}$, then

$$\sum_{t=t_0}^{t_{**}(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(\tilde{U}_{t+1})) \lesssim_{3_0} n^{-\lambda_*} u_*(n). \quad (63)$$

Proof. We set $s_t = \dim \mathcal{S}_{T_t}(\Omega)$. By (21), (22) and (37), there exists $C = C(\mathfrak{Z}_0) \geq 1$ such that

$$s_t \leq C\overline{\nu}_t =: \overline{s}_t. \quad (64)$$

By Lemma 5, there exists an operator $A_t : \mathcal{S}_{T_t}(\Omega) \rightarrow \mathbb{R}^{s_t}$ such that

$$\|A_t\|_{Y_{p,q,T_t} \rightarrow l_{\sigma_{p,q}}^{s_t}} \lesssim 1, \quad \|A_t^{-1}\|_{l_q^{s_t} \rightarrow Y_q(\Omega)} \lesssim 1. \quad (65)$$

By (9), (64) and (65),

$$\begin{aligned} & \sum_{t=t_0}^{t_{**}(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(\tilde{U}_{t+1})) \lesssim_{\mathfrak{Z}_0} \\ & \lesssim \sum_{t=t_0}^{t_{**}(n)-1} \|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} e_{k_t}(I_{\overline{s}_t} : l_{\sigma_{p,q}}^{\overline{s}_t} \rightarrow l_q^{\overline{s}_t}). \end{aligned}$$

Let $p > q$. Then $\|\cdot\|_{p,q,T_t} = \|\cdot\|_{Y_q(\tilde{U}_{t+1})}$. It was proved in [42, Step 3 in the proof of Theorem 2] that for $f \in B\hat{X}_p(\Omega)$

$$\|f - Q_t f\|_{Y_q(\tilde{U}_t)} \lesssim_{\mathfrak{Z}_0} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \overline{\nu}_t^{\frac{1}{q} - \frac{1}{p}}. \quad (66)$$

Hence,

$$\|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} \lesssim_{\mathfrak{Z}_0} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \overline{\nu}_t^{\frac{1}{q} - \frac{1}{p}}, \quad p > q. \quad (67)$$

Let $p \leq q$. It was proved in [42, Step 3 in the proof of Theorem 2] that for $f \in B\hat{X}_p(\Omega)$

$$\|f - Q_t f\|_{p,q,T_t} \lesssim_{\mathfrak{Z}_0} 2^{-\lambda_* k_* t} u_*(2^{k_* t}),$$

which implies

$$\|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} \lesssim_{\mathfrak{Z}_0} 2^{-\lambda_* k_* t} u_*(2^{k_* t}), \quad p \leq q. \quad (68)$$

Let $\varepsilon > 0$ (it will be chosen later by \mathfrak{Z}_0). Denote

$$\hat{t}(n) = t_*(n), \quad \text{if (21) holds,} \quad \text{or if (22) holds and } \lambda_* > \frac{1}{p} - \frac{1}{q},$$

and $\hat{t}(n) = t_{**}(n)$, if (22) holds and $\lambda_* < \frac{1}{p} - \frac{1}{q}$.

We set

$$k_t = \begin{cases} \lceil n \cdot 2^{-\varepsilon(t_*(n)-t)} \rceil & \text{for } t < t_*(n), \\ \lceil n \cdot 2^{-\varepsilon|t-\hat{t}(n)|} \rceil & \text{for } t_*(n) \leq t < t_{**}(n). \end{cases} \quad (69)$$

Then (58) holds.

By (21), (22), (64) and (69), there exists $l_* = l_*(\mathfrak{Z}_0) \in \mathbb{N}$ such that for sufficiently small $\varepsilon > 0$ and for any $0 \leq r < l_*$ the sequence $\left\{ \frac{k_{l_*t+r}}{s_{l_*t+r}} \right\}_{l_*t+r \leq t_*(n)-1}$ decreases not slower than some geometric progression. Moreover, for $t < t_*(n)$ we have $\frac{\bar{s}_t}{k_t} \underset{\mathfrak{Z}_0}{\lesssim} 1$. Therefore, by Theorem A,

$$e_{k_t}(I_{\bar{s}_t} : l_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow l_q^{\bar{s}_t}) \underset{p,q}{\asymp} 2^{-\frac{k_t}{\bar{s}_t} \frac{1}{s_t^q} - \frac{1}{\sigma_{p,q}}}, \quad t < t_*(n).$$

Let $p > q$. By Remark 4, we have (21). Hence,

$$\begin{aligned} & \sum_{t=t_0}^{t_*(n)-1} \|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} e_{k_t}(I_{\bar{s}_t} : l_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow l_q^{\bar{s}_t}) \underset{\mathfrak{Z}_0}{\lesssim} \\ & \lesssim 2^{-\mu_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) \bar{\nu}_{t_*(n)}^{\frac{1}{q} - \frac{1}{p}} \underset{\mathfrak{Z}_0}{\lesssim} n^{-\mu_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)). \end{aligned} \quad (67)$$

If $p \leq q$ and (21) holds, then

$$\begin{aligned} & \sum_{t=t_0}^{t_*(n)-1} \|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} e_{k_t}(I_{\bar{s}_t} : l_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow l_q^{\bar{s}_t}) \underset{\mathfrak{Z}_0}{\lesssim} \\ & \lesssim 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) \bar{\nu}_{t_*(n)}^{\frac{1}{q} - \frac{1}{p}} \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)). \end{aligned} \quad (68)$$

If $p \leq q$ and (22) holds, then

$$\begin{aligned} & \sum_{t=t_0}^{t_*(n)-1} \|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} e_{k_t}(I_{\bar{s}_t} : l_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow l_q^{\bar{s}_t}) \underset{\mathfrak{Z}_0}{\lesssim} \\ & \lesssim 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) \cdot \max_{0 \leq r \leq l_*} 2^{-\frac{k_{t_*(n)-1-r}}{\bar{s}_{t_*(n)-1-r}} \frac{1}{s_{t_*(n)-1-r}^q} - \frac{1}{p}} \lesssim_{\mathfrak{Z}_0} \\ & \lesssim 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) k_{t_*(n)-1}^{\frac{1}{q} - \frac{1}{p}} \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_*} u_*(\log n). \end{aligned} \quad (49)$$

If $p \geq q$, we get the desired estimates in Lemma since $t_*(n) = t_{**}(n)$.

Let $p < q$ (then $\sigma_{p,q} = p$). We apply Theorem A. For $t_*(n) \leq t < t_{**}(n)$ we have

$$e_{k_t}(I_{\bar{s}_t} : l_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow l_q^{\bar{s}_t}) \underset{p,q}{\lesssim} k_t^{\frac{1}{q} - \frac{1}{p}} \left(\log \left(1 + \frac{\bar{\nu}_t}{k_t} \right) \right)^{\frac{1}{p} - \frac{1}{q}} \underset{\mathfrak{Z}_0}{\lesssim} k_t^{\frac{1}{q} - \frac{1}{p}} (\log \bar{\nu}_t)^{\frac{1}{p} - \frac{1}{q}}.$$

If (21) holds, then for sufficiently small $\varepsilon > 0$

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} e_{k_t}(I_{\bar{s}_t} : l_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow l_q^{\bar{s}_t}) \underset{\mathfrak{Z}_0}{\lesssim} \quad (68), (69)$$

$$\begin{aligned}
&\lesssim \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) n^{\frac{1}{q}-\frac{1}{p}} 2^{\varepsilon(\frac{1}{p}-\frac{1}{q})(t-t_*(n))} \left(\log \left(1 + \frac{\bar{v}_t}{k_t} \right) \right)^{\frac{1}{p}-\frac{1}{q}} \quad (47), (69) \\
&\lesssim n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)).
\end{aligned}$$

If (22) holds, then

$$\begin{aligned}
&\sum_{t=t_*(n)}^{t_{**}(n)-1} \|Q_{t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,T_t}(\Omega)} e_{k_t}(I_{\bar{s}_t} : \bar{l}_{\sigma_{p,q}}^{\bar{s}_t} \rightarrow \bar{l}_q^{\bar{s}_t}) \quad (22), (46), (68), (69) \\
&\lesssim \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* t} u_*(2^t) n^{\frac{1}{q}-\frac{1}{p}} 2^{\varepsilon(\frac{1}{p}-\frac{1}{q})|t-\hat{t}(n)|} 2^{t(\frac{1}{p}-\frac{1}{q})} =: S. \\
&\text{If } \lambda_* > \frac{1}{p} - \frac{1}{q}, \text{ then } S \stackrel{(49)}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p}-\frac{1}{q}} u_*(\log n). \text{ If } \lambda_* < \frac{1}{p} - \frac{1}{q}, \text{ then } S \stackrel{(49)}{\lesssim} \\
&n^{-\lambda_*} u_*(n). \text{ This completes the proof. } \square
\end{aligned}$$

For $t \geq t_0$, $m \in \mathbb{Z}_+$ we set

$$\hat{T}_{t,m} = \{E \cap E' : E \in \tilde{T}_{t,m}, E' \in \tilde{T}_{t,m+1}, \text{mes}(E \cap E') > 0\}. \quad (70)$$

Let

$$\tilde{s}_{t,m} = \dim \mathcal{S}_{\tilde{T}_{t,m}}(\Omega), \quad s_{t,m} = \dim \mathcal{S}_{\hat{T}_{t,m}}(\Omega). \quad (71)$$

From (40) and (43) it follows that there exists $C_1(\mathfrak{Z}_0) \geq 1$ such that

$$\tilde{s}_{t,m} \leq C_1 \cdot 2^m, \quad s_{t,m} \leq C_1 \cdot 2^m. \quad (72)$$

By Lemma 5, there exists a linear isomorphism $A_{t,m} : \mathcal{S}_{\hat{T}_{t,m}}(\Omega) \rightarrow \mathbb{R}^{s_{t,m}}$ such that

$$\|A_{t,m}\|_{Y_{p,q,\hat{T}_{t,m}}(G_{m,t}) \rightarrow l_{\sigma_{p,q}}^{s_{t,m}}} \lesssim 1, \quad \|A_{t,m}^{-1}\|_{l_q^{s_{t,m}} \rightarrow Y_q(G_{m,t})} \lesssim 1. \quad (73)$$

Lemma 7. *There exists a sequence $\{k_{t,m}\}_{t_0 \leq t < t_*(n), m \in \mathbb{Z}_+} \subset \mathbb{N}$ such that $\sum_{t_0 \leq t < t_*(n), m \in \mathbb{Z}_+} (k_{t,m} -$*

1) $\lesssim n$ and the following assertions hold.

1. *Let (21) hold, let the sequence $\{k_t\}_{t \geq t_0}$ be defined by (69), and let $\sigma_*(n)$ be such as in Theorem 4. Then for $p > q$*

$$\sum_{t=t_0}^{t_*(n)-1} \sum_{m=m_t+1}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) +$$

$$+ \sum_{t=t_0}^{t_*(n)-1} e_{k_t}(P_{t,m_t+1} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\min\{\delta_*, \mu_*\beta_*\} + \frac{1}{q} - \frac{1}{p}} \sigma_*(n),$$

and for $p \leq q$

$$\sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \rightarrow Y_q(G_{m,t})) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\min\{\delta_*, \lambda_*\beta_*\} + \frac{1}{q} - \frac{1}{p}} \sigma_*(n).$$

2. Suppose that (22) holds. Then

$$\sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \rightarrow Y_q(G_{m,t})) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p} - \frac{1}{q}} u_*(\log n).$$

Proof. By (9) and (73), it suffices to estimate

$$\begin{aligned} & \sum_{t=t_0}^{t_*(n)-1} \sum_{m=m_t+1}^{\infty} \|P_{t,m+1} - P_{t,m}\|_{\hat{X}_p(\Omega) \rightarrow Y_q(G_t)} e_{k_{t,m}}(I_{s_{t,m}} : l_q^{s_{t,m}} \rightarrow l_q^{s_{t,m}}) + \\ & + \sum_{t=t_0}^{t_*(n)-1} \|P_{t,m_t+1} - Q_t\|_{\hat{X}_p(\Omega) \rightarrow Y_q(G_t)} e_{k_t}(I_{\tilde{s}_{t,m_t+1}} : l_q^{\tilde{s}_{t,m_t+1}} \rightarrow l_q^{\tilde{s}_{t,m_t+1}}) =: S_1 \end{aligned} \quad (74)$$

for $p > q$ and

$$\sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} \|P_{t,m+1} - P_{t,m}\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,\tilde{T}_{t,m}}(\Omega)} e_{k_{t,m}}(I_{s_{t,m}} : l_p^{s_{t,m}} \rightarrow l_q^{s_{t,m}}) =: S_2 \quad (75)$$

for $p \leq q$.

Let

$$\bar{s}_{t,m} = \lceil C_1 \cdot 2^m \rceil. \quad (76)$$

We define the number $t_1(n)$ as follows. In assertion 1 of Lemma we set

$$t_1(n) = \begin{cases} t_0 & \text{if } \delta_* < \mu_*\beta_*, \\ t_*(n), & \text{if } \delta_* > \mu_*\beta_*. \end{cases} \quad (77)$$

for $p > q$ and

$$t_1(n) = \begin{cases} t_0, & \text{if } \delta_* < \lambda_*\beta_*, \\ t_*(n), & \text{if } \delta_* > \lambda_*\beta_*. \end{cases} \quad (78)$$

for $p \leq q$. In assertion 2 we set

$$t_1(n) = t_*(n). \quad (79)$$

Denote

$$m_t^* = \lceil \log(n \cdot 2^{-\varepsilon|t-t_1(n)|}) \rceil, \quad k_{t,m} = \lceil n \cdot 2^{-\varepsilon(|t-t_1(n)|+|m-m_t^*|)} \rceil. \quad (80)$$

Then $k_{t,m} \in \mathbb{N}$ and $\sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} (k_{t,m} - 1) \lesssim_{30,\varepsilon} n$.

Case $p > q$. From Hölder's inequality it follows that for $f \in B\hat{X}_p(\Omega)$, $m > m_t$

$$\begin{aligned} & \|P_{t,m+1}f - P_{t,m}f\|_{Y_q(G_{m,t})} \stackrel{(42)}{\lesssim_{30}} \\ & \lesssim 2^{-\mu_*k_*t-\delta_*(m-m_t)} u_*(2^{k_*t}) (\text{card } \hat{T}_{t,m})^{\frac{1}{q}-\frac{1}{p}} \stackrel{(40),(43)}{\lesssim_{30}} 2^{-\mu_*k_*t-\delta_*(m-m_t)} u_*(2^{k_*t}) \cdot 2^{m(\frac{1}{q}-\frac{1}{p})} \stackrel{(39)}{\lesssim_{30}} \\ & \lesssim 2^{-\mu_*k_*t} u_*(2^{k_*t}) \cdot \bar{\nu}_t^{\delta_*} \cdot 2^{m(-\delta_*+\frac{1}{q}-\frac{1}{p})}. \end{aligned}$$

Further, for $f \in B\hat{X}_p(\Omega)$ by Hölder's inequality we get

$$\begin{aligned} & \|P_{t,m_t+1}f - Q_tf\|_{Y_q(G_t)} \leq \|f - P_{t,m_t+1}f\|_{Y_q(G_t)} + \|f - Q_tf\|_{Y_q(G_t)} \stackrel{(40),(42),(66)}{\lesssim_{30}} \\ & \lesssim 2^{-\mu_*k_*t} u_*(2^{k_*t}) \cdot 2^{m_t(\frac{1}{q}-\frac{1}{p})} + 2^{-\mu_*k_*t} u_*(2^{k_*t}) \bar{\nu}_t^{\frac{1}{q}-\frac{1}{p}} \stackrel{(39)}{\lesssim_{30}} 2^{-\mu_*k_*t} u_*(2^{k_*t}) \bar{\nu}_t^{\frac{1}{q}-\frac{1}{p}}. \end{aligned}$$

Thus,

$$\|P_{t,m+1}f - P_{t,m}f\|_{Y_q(G_{m,t})} \lesssim_{30} 2^{-\mu_*k_*t} u_*(2^{k_*t}) \cdot \bar{\nu}_t^{\delta_*} \cdot 2^{m(-\delta_*+\frac{1}{q}-\frac{1}{p})}, \quad (81)$$

$$\|P_{t,m_t+1}f - Q_tf\|_{Y_q(G_t)} \lesssim_{30} 2^{-\mu_*k_*t} u_*(2^{k_*t}) \bar{\nu}_t^{\frac{1}{q}-\frac{1}{p}}. \quad (82)$$

From (76) we get that $\bar{s}_{t,m_t+1} = \lceil C_1 \cdot 2^{m_t+1} \rceil \stackrel{(39)}{\leq} \lceil 4C_1 \bar{\nu}_t \rceil$. Similarly as in Lemma 6 we prove that

$$\begin{aligned} & \sum_{t=t_0}^{t_*(n)-1} 2^{-\mu_*k_*t} u_*(2^{k_*t}) \bar{\nu}_t^{\frac{1}{q}-\frac{1}{p}} e_{k_t}(I_{\bar{s}_{t,m_t+1}} : l_q^{\bar{s}_{t,m_t+1}} \rightarrow l_q^{\bar{s}_{t,m_t+1}}) \stackrel{(72),(76)}{\leq} \\ & \leq \sum_{t=t_0}^{t_*(n)-1} 2^{-\mu_*k_*t} u_*(2^{k_*t}) \bar{\nu}_t^{\frac{1}{q}-\frac{1}{p}} e_{k_t}(I_{\bar{s}_{t,m_t+1}} : l_q^{\bar{s}_{t,m_t+1}} \rightarrow l_q^{\bar{s}_{t,m_t+1}}) \lesssim_{30} \\ & \lesssim n^{-\mu_*\beta_*+\frac{1}{q}-\frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)). \end{aligned} \quad (83)$$

Let $\varepsilon > 0$ be sufficiently small. From Theorem A it follows that for $m \leq m_t^*$

$$e_{k_{t,m}}(I_{s_{t,m}} : l_q^{s_{t,m}} \rightarrow l_q^{s_{t,m}}) \stackrel{(72),(76)}{\leq} e_{k_{t,m}}(I_{\bar{s}_{t,m}} : l_q^{\bar{s}_{t,m}} \rightarrow l_q^{\bar{s}_{t,m}}) \stackrel{(76),(80)}{\lesssim_{p,q}} 2^{-\frac{k_{t,m}}{\bar{s}_{t,m}}}.$$

In addition, there exists $l_* = l_*(\mathfrak{Z}_0)$ such that for any $t_0 \leq t < t_*(n)$, $0 \leq r < l_*$ the sequence $\left\{ \frac{k_{t,l_*m+r}}{\bar{s}_{t,l_*m+r}} \right\}_{l_*m+r \leq m_t^*}$ decreases not slower than some geometric progression. If $m > m_t^*$, then

$$e_{k_t,m}(I_{s_t,m} : l_q^{s_t,m} \rightarrow l_q^{s_t,m}) \stackrel{(72),(76)}{\leq} e_{k_t,m}(I_{\bar{s}_t,m} : l_q^{\bar{s}_t,m} \rightarrow l_q^{\bar{s}_t,m}) \underset{p,q}{\lesssim} 1.$$

By Remark 4, (21) holds. Hence,

$$\begin{aligned} S_1 &\stackrel{(74),(81),(82)}{\underset{\mathfrak{Z}_0}{\lesssim}} \sum_{t=t_0}^{t_*(n)-1} \sum_{m=m_t+1}^{\infty} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \cdot \bar{\nu}_t^{\delta_*} \cdot 2^{m(-\delta_* + \frac{1}{q} - \frac{1}{p})} e_{k_t,m}(I_{s_t,m} : l_q^{s_t,m} \rightarrow l_q^{s_t,m}) + \\ &\quad + \sum_{t=t_0}^{t_*(n)-1} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \bar{\nu}_t^{\frac{1}{q} - \frac{1}{p}} e_{k_t}(I_{\bar{s}_t, m_t+1} : l_{\sigma_{p,q}}^{\bar{s}_t, m_t+1} \rightarrow l_q^{\bar{s}_t, m_t+1}) \stackrel{(29),(83)}{\underset{\mathfrak{Z}_0, \varepsilon}{\lesssim}} \\ &\lesssim \sum_{t=t_0}^{t_*(n)-1} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \cdot \bar{\nu}_t^{\delta_*} \cdot 2^{m_t^*(-\delta_* + \frac{1}{q} - \frac{1}{p})} + n^{-\mu_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)) \stackrel{(21),(80)}{\underset{\mathfrak{Z}_0, \varepsilon}{\lesssim}} \\ &\lesssim \sum_{t=t_0}^{t_*(n)-1} 2^{-\mu_* k_* t} u_*(2^{k_* t}) \cdot 2^{\delta_* \gamma_* k_* t} \psi_*^{\delta_*}(2^{k_* t}) \cdot (n \cdot 2^{-\varepsilon |t-t_1(n)|})^{-\delta_* + \frac{1}{q} - \frac{1}{p}} + \\ &\quad + n^{-\mu_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)) =: S'_1. \end{aligned}$$

Recall that $\beta_* = \frac{1}{\gamma_*}$. If $\delta_* < \mu_* \beta_*$, then for sufficiently small $\varepsilon > 0$ we have $S'_1 \stackrel{(77)}{\underset{\mathfrak{Z}_0}{\lesssim}} n^{-\delta_* + \frac{1}{q} - \frac{1}{p}}$. If $\delta_* > \mu_* \beta_*$, then for sufficiently small $\varepsilon > 0$ we get

$$\begin{aligned} S'_1 &\stackrel{(77)}{\underset{\mathfrak{Z}_0}{\lesssim}} 2^{-\mu_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) \bar{\nu}_{t_*(n)}^{\delta_*} \cdot n^{-\delta_* + \frac{1}{q} - \frac{1}{p}} + n^{-\mu_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\mu_*}(n) u_*(n^{\beta_*} \varphi_*(n)) \stackrel{(47)}{\underset{\mathfrak{Z}_0}{\lesssim}} \\ &\lesssim n^{-\mu_* \beta_* + \frac{1}{q} - \frac{1}{p}} \sigma_*(n). \end{aligned}$$

This completes the proof for $p > q$.

Case $p \leq q$. Let $f \in B\hat{X}_p(\Omega)$. Then for $m \leq m_t$

$$\|P_{t,m+1}f - P_{t,m}f\|_{p,q,\tilde{T}_{t,m}} \stackrel{(41)}{\underset{\mathfrak{Z}_0}{\lesssim}} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \leq 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_*(m-m_t)}, \quad (84)$$

and for $m > m_t$

$$\|P_{t,m+1}f - P_{t,m}f\|_{p,q,\tilde{T}_{t,m}} \stackrel{(18),(42)}{\underset{\mathfrak{Z}_0}{\lesssim}} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_*(m-m_t)}. \quad (85)$$

By Theorem A, we have for $m \leq m_t^*$

$$e_{k_{t,m}}(I_{s_{t,m}} : l_p^{s_{t,m}} \rightarrow l_q^{s_{t,m}}) \stackrel{(72),(76)}{\leq} e_{k_{t,m}}(I_{\bar{s}_{t,m}} : \bar{l}_p^{\bar{s}_{t,m}} \rightarrow \bar{l}_q^{\bar{s}_{t,m}}) \stackrel{(76),(80)}{\lesssim_{p,q}} 2^{-\frac{k_{t,m}}{\bar{s}_{t,m}} \frac{1}{q} - \frac{1}{p}}$$

(moreover, there exists $l_* = l_*(\mathfrak{Z}_0) \in \mathbb{N}$ such that for any $0 \leq r < l_*$, $t_0 \leq t < t_*(n)$ the sequence $\left\{ \frac{k_{t,l_*m+r}}{\bar{s}_{t,l_*m+r}} \right\}_{0 \leq l_*m+r \leq m_t^*}$ decreases not slower than some geometric progression), and for $m > m_t^*$ we have

$$e_{k_{t,m}}(I_{s_{t,m}} : l_p^{s_{t,m}} \rightarrow l_q^{s_{t,m}}) \lesssim_{p,q} \min \left\{ \frac{\log \left(1 + \frac{\bar{s}_{t,m}}{k_{t,m}} \right)}{k_{t,m}}, 1 \right\}^{\frac{1}{p} - \frac{1}{q}}.$$

This implies that

$$\begin{aligned} S_2 &\stackrel{(75),(84),(85)}{\lesssim_{\mathfrak{Z}_0}} \sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{m_t^*} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_*(m-m_t)} 2^{-\frac{k_{t,m}}{\bar{s}_{t,m}} \frac{1}{q} - \frac{1}{p}} + \\ &+ \sum_{t=t_0}^{t_*(n)-1} \sum_{m=m_t^*+1}^{\infty} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_*(m-m_t)} \min \left\{ \frac{\log \left(1 + \frac{\bar{s}_{t,m}}{k_{t,m}} \right)}{k_{t,m}}, 1 \right\}^{\frac{1}{p} - \frac{1}{q}} \stackrel{(39),(76),(80)}{\lesssim_{\mathfrak{Z}_0}} \\ &\lesssim \sum_{t=t_0}^{t_*(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_* m_t^*} \bar{\nu}_t^{\delta_*} k_{t,m_t^*}^{\frac{1}{q} - \frac{1}{p}} \stackrel{(80)}{\lesssim_{\mathfrak{Z}_0}} \\ &\lesssim \sum_{t=t_0}^{t_*(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) (n \cdot 2^{-\varepsilon |t-t_1(n)|})^{-\delta_* + \frac{1}{q} - \frac{1}{p}} \bar{\nu}_t^{\delta_*} =: S'_2. \end{aligned}$$

If (21) holds, then $\bar{\nu}_t = 2^{\gamma_* k_* t} \psi_*(2^{k_* t})$. Hence, for $\delta_* < \lambda_* \beta_*$ we have $S'_2 \stackrel{(78)}{\lesssim_{\mathfrak{Z}_0}} n^{-\delta_* + \frac{1}{q} - \frac{1}{p}}$, and for $\delta_* > \lambda_* \beta_*$ we get

$$S'_2 \stackrel{(78)}{\lesssim_{\mathfrak{Z}_0}} 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) n^{-\delta_* + \frac{1}{q} - \frac{1}{p}} \bar{\nu}_{t_*(n)}^{\delta_*} \stackrel{(47)}{\lesssim_{\mathfrak{Z}_0}} n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} \sigma_*(n).$$

If (22) holds, then $\bar{\nu}_t = 2^{\gamma_* 2^t} \psi_*(2^{2^t})$, $k_* = 1$ and

$$S'_2 \stackrel{(44),(49),(79)}{\lesssim_{\mathfrak{Z}_0}} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_*} u_*(\log n) \leq n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p} - \frac{1}{q}} u_*(\log n).$$

This completes the proof. \square

Theorems 4 and 5 for $p \geq q$ follow from (8), (45), (50), (51), (52) and Lemmas 4, 6, 7.

Consider the case $p < q$.

We set

$$t_2(n) = \begin{cases} t_*(n), & \text{if (21) holds, or if (22) holds and } \lambda_* > \frac{1}{p} - \frac{1}{q}, \\ t_{**}(n), & \text{if (22) holds and } \lambda_* < \frac{1}{p} - \frac{1}{q}. \end{cases} \quad (86)$$

Lemma 8. *Let $p < q$. We set*

$$k_{t,m} = \lceil n \cdot 2^{-\varepsilon|t-t_2(n)|-\varepsilon(m-m_t)} \rceil, \quad t_*(n) \leq t < t_{**}(n), \quad m \geq m_t. \quad (87)$$

Then

$$\sum_{t_*(n) \leq t < t_{**}(n), m \geq m_t} (k_{t,m} - 1) \underset{3_0, \varepsilon}{\lesssim} n \quad (88)$$

and for sufficiently small $\varepsilon > 0$ the following assertions hold.

1. *Let (21) hold. Then*

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{m=m_t}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{3_0}{\lesssim} n^{-\lambda_*\beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)).$$

2. *Let (22) hold. Then*

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{m=m_t}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{3_0}{\lesssim} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p} - \frac{1}{q}} u_*(\log n)$$

for $\lambda_* > \frac{1}{p} - \frac{1}{q}$,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{m=m_t}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{3_0}{\lesssim} n^{-\lambda_*} u_*(n)$$

for $\lambda_* < \frac{1}{p} - \frac{1}{q}$.

Proof. The relation (88) follows from (87).

Let $\hat{T}_{t,m}$, $s_{t,m}$ be defined by (70), (71). From (72) it follows that $s_{t,m} \leq C_1 \cdot 2^m$. We set $\bar{s}_{t,m} = \lceil C_1 \cdot 2^m \rceil$. Then

$$\bar{s}_{t,m_t} \underset{3_0}{\overset{(39)}{\lesssim}} \bar{\nu}_t. \quad (89)$$

By (9) and (73), it suffices to estimate

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{m=m_t}^{\infty} \|P_{t,m+1} - P_{t,m}\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,\hat{T}_{t,m}}(G_t)} e_{k_{t,m}}(I_{\bar{s}_{t,m}} : l_p^{\bar{s}_{t,m}} \rightarrow l_q^{\bar{s}_{t,m}}) =: S.$$

From (18), (41), (42) it follows that

$$\|P_{t,m+1} - P_{t,m}\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,\hat{T}_{t,m}}(G_t)} \lesssim_{3_0} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot 2^{-\delta_*(m-m_t)}.$$

Since $\bar{s}_{t,m} \stackrel{(76)}{\geq} C_1 \cdot 2^{m_t} \stackrel{(39)}{\gtrsim_{3_0}} \bar{\nu}_t \stackrel{(21),(22)}{\gtrsim_{3_0}} \bar{\nu}_{t_*(n)} \stackrel{(44)}{\geq} n$ and $k_{t,m} \stackrel{(87)}{\leq} n$, we get by Theorem A that

$$e_{k_{t,m}}(I_{\bar{s}_{t,m}} : l_p^{\bar{s}_{t,m}} \rightarrow l_q^{\bar{s}_{t,m}}) \lesssim_{3_0} \min \left\{ \frac{\log \left(1 + \frac{\bar{s}_{t,m}}{k_{t,m}} \right)}{k_{t,m}}, 1 \right\}^{\frac{1}{p} - \frac{1}{q}}.$$

Hence, for sufficiently small $\varepsilon > 0$ we have

$$\begin{aligned} S &\lesssim_{3_0} \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \left\{ \frac{\log \left(1 + \frac{\bar{s}_{t,m_t}}{k_{t,m_t}} \right)}{k_{t,m_t}} \right\}^{\frac{1}{p} - \frac{1}{q}} \stackrel{(87),(89)}{\lesssim_{3_0}} \\ &\lesssim \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \cdot n^{\frac{1}{q} - \frac{1}{p}} \cdot 2^{\varepsilon |t-t_2(n)|(\frac{1}{p} - \frac{1}{q})} \left(\log \left(1 + \frac{\bar{\nu}_t}{\lceil n \cdot 2^{-\varepsilon |t-t_2(n)|} \rceil} \right) \right)^{\frac{1}{p} - \frac{1}{q}}. \end{aligned}$$

This together with (44), (47), (49), (86) yields the desired estimates for sufficiently small $\varepsilon > 0$. \square

Lemma 9. *Let $p < q$. Then there exists a sequence $\{\tilde{k}_t\}_{t_*(n) \leq t < t_{**}(n)} \subset \mathbb{N}$ such that*

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} (\tilde{k}_t - 1) \lesssim_{3_0} n \tag{90}$$

and the following assertions hold.

1. *Let (21) hold. Then*

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} e_{\tilde{k}_t}(P_{t,m_t} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \lesssim_{3_0} n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} u_*(n^{\beta_*} \varphi_*(n)) \varphi_*^{-\lambda_*}(n).$$

2. Let (22) hold. Then for $\lambda_* > \frac{1}{p} - \frac{1}{q}$ we have

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} e_{\tilde{k}_t}(P_{t,m_t} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}} (\log n)^{-\lambda_*+\frac{1}{p}-\frac{1}{q}} u_*(\log n),$$

and for $\lambda_* < \frac{1}{p} - \frac{1}{q}$ we have

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} e_{\tilde{k}_t}(P_{t,m_t} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_*} u_*(n).$$

We shall use Theorem C and the following assertion.

Lemma 10. (see [38]). Let (\mathcal{T}, ξ_*) be a tree with finite vertex set, let

$$\text{card } \mathbf{V}_1(\xi) \leq k, \quad \xi \in \mathbf{V}(\mathcal{T}), \quad (91)$$

and let the mapping $\Phi : 2^{\mathbf{V}(\mathcal{T})} \rightarrow \mathbb{R}_+$ satisfy the following condition:

$$\Phi(V_1 \cup V_2) \geq \Phi(V_1) + \Phi(V_2), \quad V_1, V_2 \subset \mathbf{V}(\mathcal{T}), \quad V_1 \cap V_2 = \emptyset, \quad (92)$$

$\Phi(\mathbf{V}(\mathcal{T})) > 0$. Then there is a number $C(k) > 0$ such that for any $n \in \mathbb{N}$ there exists a partition \mathfrak{S}_n of the tree \mathcal{T} into at most $C(k)n$ subtrees \mathcal{T}_j , which satisfies the following conditions:

1. $\Phi(\mathbf{V}(\mathcal{T}_j)) \leq \frac{(k+2)\Phi(\mathbf{V}(\mathcal{T}))}{n}$ for any j such that $\text{card } \mathbf{V}(\mathcal{T}_j) \geq 2$;
2. if $m \leq 2n$, then each element of \mathfrak{S}_n intersects with at most $C(k)$ elements of \mathfrak{S}_m .

Proof of Lemma 9. Step 1. We define the numbers $t_2(n)$ by (86); $\varepsilon = \varepsilon(\mathfrak{Z}_0)$ will be chosen later. The sequence $\tilde{k}_t(n)$ is defined so that

$$\tilde{k}_t(n) - 1 \underset{\mathfrak{Z}_0}{\lesssim} n \cdot 2^{-\varepsilon|t-t_2(n)|} \quad \text{if (21) holds,} \quad (93)$$

$$\tilde{k}_t(n) - 1 \underset{\mathfrak{Z}_0}{\lesssim} \max \{ n \cdot 2^{-\varepsilon|t-t_2(n)|}, 2^t \} \quad \text{if (22) holds.} \quad (94)$$

This together with (49) implies (90).

Further we consider $t_*(n) \leq t < t_{**}(n)$.

Step 2. Let \mathbf{T} be a partition of Γ_t into subtrees. Then $\mathbf{T} = \{\mathcal{A}_{t,i,s}\}_{i \in \hat{J}_t, s \in I_{t,i}}$, where $\{\mathcal{A}_{t,i,s}\}_{s \in I_{t,i}}$ — is a partition of $\mathcal{A}_{t,i}$. Denote by $\hat{\xi}_{t,i,s}$ the root of $\mathcal{A}_{t,i,s}$.

Define the operator $\hat{P}_{\mathbf{T}} : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)$ as follows. We set $\hat{P}_{\mathbf{T}} f|_{\Omega \setminus G_t} = 0$,

$$\hat{P}_{\mathbf{T}} f|_{\Omega_{\mathcal{A}_{t,i,s}}} = 0, \quad \text{if } \hat{\xi}_{t,i,s} = \hat{\xi}_{t,i} \quad (95)$$

$$\hat{P}_{\mathbf{T}}f|_{\Omega_{\mathcal{A}_{t,i,s}}} = (P_{t,m_t}f - Q_tf)|_{\Omega_{\mathcal{A}_{t,i,s}}} \quad \text{if } \mathbf{V}(\mathcal{A}_{t,i,s}) = \{\hat{\xi}_{t,i,s}\}; \quad (96)$$

in other cases we set

$$\hat{P}_{\mathbf{T}}f|_{\Omega_{\mathcal{A}_{t,i,s}}} = (P_{\Omega_{\hat{\xi}_{t,i,s}}}f - Q_tf)|_{\Omega_{\mathcal{A}_{t,i,s}}} \quad (97)$$

(see Assumption 1). Let $T = \{\Omega_{\mathcal{A}'}\}_{\mathcal{A}' \in \mathbf{T}}$. Notice that

$$\hat{P}_{\mathbf{T}}f \in \mathcal{S}_T(\Omega). \quad (98)$$

Let $f \in B\hat{X}_p(\Omega)$. If $\hat{\xi}_{t,i,s} = \hat{\xi}_{t,i}$, then

$$\|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}}f\|_{Y_q(\Omega_{\mathcal{A}_{t,i,s}})} \stackrel{(34),(36),(41),(95)}{\lesssim_{30}} 2^{-\lambda_*k_*t}u_*(2^{k_*t})\|f\|_{X_p(\Omega_{\mathcal{A}_{t,i,s}})}. \quad (99)$$

If $\mathbf{V}(\mathcal{A}_{t,i,s}) = \{\hat{\xi}_{t,i,s}\}$, then

$$\|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}}f\|_{Y_q(\Omega_{\mathcal{A}_{t,i,s}})} \stackrel{(96)}{=} 0. \quad (100)$$

In other cases

$$\|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}}f\|_{Y_q(\Omega)} \stackrel{(15),(19),(41),(97)}{\lesssim_{30}} 2^{-\lambda_*k_*t}u_*(2^{k_*t})\|f\|_{X_p(\Omega_{\mathcal{A}_{t,i,s}})}. \quad (101)$$

We set $\mathbf{T}' = \{\mathcal{A}' \in \mathbf{T} : \text{card } \mathbf{V}(\mathcal{A}') \geq 2\}$. Then for any $f \in B\hat{X}_p(\Omega)$

$$\|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}}f\|_{Y_q(G_t)} \stackrel{(99),(100),(101)}{\lesssim_{30}} 2^{-\lambda_*k_*t}u_*(2^{k_*t}) \left(\sum_{\mathcal{A}' \in \mathbf{T}'} \|f\|_{X_p(\Omega_{\mathcal{A}'})}^q \right)^{1/q}. \quad (102)$$

Step 3. Let $r \in \mathbb{N}$,

$$r \leq \frac{\bar{\nu}_t}{4}. \quad (103)$$

Denote by \mathcal{N}_r the family of partitions \mathbf{T} of the graph Γ_t into subtrees such that $\text{card } \{\mathcal{A}_{t,i,s} \in \mathbf{T} : \hat{\xi}_{t,i,s} \neq \hat{\xi}_{t,i}\} \leq r$. The number $|\mathcal{N}_r|$ can be estimated from above by the number of choices of sets of vertices $\hat{\xi}_{t,i,s} \neq \hat{\xi}_{t,i}$ in $\mathbf{V}(\Gamma_t)$. Therefore,

$$|\mathcal{N}_r| \leq \sum_{m=0}^r C_{|\mathbf{V}(\Gamma_t)|}^m \stackrel{(21),(22)}{\leq} \sum_{m=0}^r C_{\lceil c_3 \bar{\nu}_t \rceil}^m \stackrel{(103)}{\lesssim} C_{\lceil c_3 \bar{\nu}_t \rceil}^r \leq \left(\frac{e \lceil c_3 \bar{\nu}_t \rceil}{r} \right)^r. \quad (104)$$

If (21) holds, then we set

$$r_t = \lceil n \cdot 2^{-2\varepsilon|t-t_2(n)|-c} \rceil, \quad n \geq N(\mathfrak{Z}_0), \quad (105)$$

where $c = c(\mathfrak{Z}_0)$, $N(\mathfrak{Z}_0)$ are such that $r_t \leq \frac{\bar{v}_t}{4}$ for $n \geq N(\mathfrak{Z}_0)$. Then in the case $n \cdot 2^{-2\varepsilon|t-t_2(n)|-c} \geq 1$ we have

$$\begin{aligned}
\log |\mathcal{N}_{r_{t-1}}| &\leq \log |\mathcal{N}_{r_t}| \stackrel{(104)}{\lesssim_{\mathfrak{Z}_0}} r_t \log \frac{c_3 e \bar{v}_t}{r_t} \stackrel{(86),(105)}{\lesssim_{\mathfrak{Z}_0}} \\
&\lesssim n \cdot 2^{-2\varepsilon(t-t_*(n))} \log \left(\frac{c_3 e \bar{v}_{t_*(n)}}{n \cdot 2^{-2\varepsilon(t-t_*(n))-c}} \cdot \frac{\bar{v}_t}{\bar{v}_{t_*(n)}} \right) \stackrel{(21),(46),(47)}{\lesssim_{\mathfrak{Z}_0}} \\
&\lesssim n \cdot 2^{-2\varepsilon(t-t_*(n))} \log(2^{2\varepsilon(t-t_*(n))} \cdot 2^{2\gamma_* k_*(t-t_*(n))} \cdot e) \stackrel{\lesssim_{\mathfrak{Z}_0}}{\lesssim} \\
&\lesssim n \cdot 2^{-2\varepsilon(t-t_*(n))} \cdot (2\gamma_* k_* + 2\varepsilon)(t - t_*(n) + 1) \stackrel{\lesssim_{\mathfrak{Z}_0, \varepsilon}}{\lesssim} n \cdot 2^{-\varepsilon(t-t_*(n))};
\end{aligned}$$

if $n \cdot 2^{-2\varepsilon|t-t_2(n)|-c} < 1$, then $|\mathcal{N}_{r_{t-1}}| \stackrel{(104)}{=} C_{|\mathbf{V}(\Gamma_t)|}^0 = 1$ and $\log |\mathcal{N}_{r_{t-1}}| = 0$. Therefore,

$$\log |\mathcal{N}_{r_{t-1}}| \stackrel{\lesssim_{\mathfrak{Z}_0, \varepsilon}}{\lesssim} n \cdot 2^{-\varepsilon(t-t_*(n))}. \quad (106)$$

If (22) holds, then we set

$$r_t = \left\lceil \frac{n \cdot 2^{-\varepsilon|t-t_2(n)|}}{2^{t+c}} \right\rceil, \quad (107)$$

where $c = c(\mathfrak{Z}_0)$ is such that $r_t \leq \frac{\bar{v}_t}{4}$. Then

$$\begin{aligned}
\log |\mathcal{N}_{r_{t-1}}| &\leq \log |\mathcal{N}_{r_t}| \stackrel{(104)}{\lesssim_{\mathfrak{Z}_0}} r_t \log \frac{e c_3 \bar{v}_t}{r_t} \leq r_t \log(c_3 e \bar{v}_t) \stackrel{(22),(46)}{\lesssim_{\mathfrak{Z}_0}} \\
&\lesssim r_t \cdot 2^t \lesssim_{\mathfrak{Z}_0} \max \{n \cdot 2^{-\varepsilon|t-t_2(n)|}, 2^t\};
\end{aligned}$$

i.e.,

$$\log |\mathcal{N}_{r_{t-1}}| \stackrel{\lesssim_{\mathfrak{Z}_0, \varepsilon}}{\lesssim} \max \{n \cdot 2^{-\varepsilon|t-t_2(n)|}, 2^t\}. \quad (108)$$

Step 4. Consider the tree $\tilde{\mathcal{A}}_t$ with vertex set $\mathbf{V}(\mathcal{A}) \setminus \mathbf{V}(\tilde{\Gamma}_{t+1})$. For $f \in B\hat{X}_p(\Omega)$ we define the function $\Phi_f : 2^{\mathbf{V}(\tilde{\mathcal{A}}_t)} \rightarrow \mathbb{R}_+$ by

$$\Phi_f(\mathbf{W}) = \sum_{\xi \in \mathbf{W} \cap \mathbf{V}(\Gamma_t)} \|f\|_{X_p(\hat{F}(\xi))}^p. \quad (109)$$

Then $\Phi_f(\mathbf{W}_1 \sqcup \mathbf{W}_2) = \Phi_f(\mathbf{W}_1) + \Phi_f(\mathbf{W}_2)$. From (11) and Lemma 10 it follows that there exists a sequence of partitions $\{\mathbf{T}_{f,t,l}\}_{0 \leq l \leq \log r_t}$ of the tree $\tilde{\mathcal{A}}_t$ such that

$$\text{card } \mathbf{T}_{f,t,l} \leq r_t \cdot 2^{-l}, \quad (110)$$

$$\text{card}\{\mathcal{A}'' \in \mathbf{T}_{f,t,l\pm 1} : \mathbf{V}(\mathcal{A}'') \cap \mathbf{V}(\mathcal{A}') \neq \emptyset\} \underset{30}{\lesssim} 1, \quad \mathcal{A}' \in \mathbf{T}_{f,t,l}, \quad (111)$$

and for any subtree $\mathcal{A}' \in \mathbf{T}_{f,t,l}$ such that $\text{card } \mathbf{V}(\mathcal{A}') \geq 2$ the following estimate holds:

$$\Phi_f(\mathbf{V}(\mathcal{A}')) \underset{30}{\lesssim} \frac{2^l}{r_t}. \quad (112)$$

Here we may assume that

$$\text{card } \mathbf{T}_{f,t, \lfloor \log r_t \rfloor} = 1. \quad (113)$$

Denote $\mathbf{T}_{f,t,l}^* := \mathbf{T}_{f,t,l}|_{\Gamma_t}$.

We have

$$\mathbf{T}_{f,t,0}^* = \mathbf{T}_{f,t,0}|_{\Gamma_t} \in \mathcal{N}_{r_{t-1}}, \quad (114)$$

$$\sup_{f \in B\hat{X}_p(\Omega)} \|P_{t,m_t}f - Q_t f - \hat{P}_{\mathbf{T}_{f,t,0}^*} f\|_{Y_q(G_t)} \stackrel{(102),(109),(110),(112)}{\underset{30}{\lesssim}} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) r_t^{\frac{1}{q} - \frac{1}{p}} =: A_t. \quad (115)$$

Step 5. Let (21) hold. Then for sufficiently small $\varepsilon > 0$ we get

$$\begin{aligned} \sum_{t=t_*(n)}^{t_{**}(n)-1} A_t &\stackrel{(86),(105),(115)}{\underset{30}{\lesssim}} \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) (n \cdot 2^{-2\varepsilon(t-t_*(n))})^{\frac{1}{q} - \frac{1}{p}} \stackrel{(47)}{\underset{30}{\lesssim}} \\ &\lesssim n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} u_*(n^{\beta_*} \varphi_*(n)) \varphi_*^{-\lambda_*}(n); \end{aligned}$$

i.e.,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} A_t \underset{30}{\lesssim} n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} u_*(n^{\beta_*} \varphi_*(n)) \varphi_*^{-\lambda_*}(n). \quad (116)$$

Let (22) hold. Then

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} A_t \stackrel{(107),(115)}{\underset{30}{\lesssim}} \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* t} u_*(2^t) (n \cdot 2^{-\varepsilon|t-t_2(n)|-t})^{\frac{1}{q} - \frac{1}{p}} =: A.$$

In $\lambda_* > \frac{1}{p} - \frac{1}{q}$, then for sufficiently small $\varepsilon > 0$ we have

$$A \stackrel{(86)}{\underset{30}{\lesssim}} 2^{-\lambda_* t_*(n)} u_*(2^{t_*(n)}) \cdot n^{\frac{1}{q} - \frac{1}{p}} \cdot 2^{\left(\frac{1}{p} - \frac{1}{q}\right)t_*(n)} \stackrel{(49)}{\underset{30}{\lesssim}} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p} - \frac{1}{q}} u_*(\log n);$$

i.e.,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} A_t \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}} (\log n)^{-\lambda_*+\frac{1}{p}-\frac{1}{q}} u_*(\log n). \quad (117)$$

If $\lambda_* < \frac{1}{p} - \frac{1}{q}$, then for sufficiently small $\varepsilon > 0$

$$A \underset{\mathfrak{Z}_0}{\lesssim}^{(86)} 2^{-\lambda_* t_{**}(n)} u_*(2^{t_{**}(n)}) \cdot n^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{(\frac{1}{p}-\frac{1}{q})t_{**}(n)} \underset{\mathfrak{Z}_0}{\lesssim}^{(49)} n^{-\lambda_*} u_*(n);$$

i.e.,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} A_t \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_*} u_*(n). \quad (118)$$

Step 6. Let $0 \leq l \leq \log r_t$, $\mathbf{T}_{t,l} = \mathbf{T}_{\hat{f},t,l}^*$ for some function $\hat{f} \in B\hat{X}_p(\Omega)$. We set

$$\hat{k}_{t,l} = \lceil n \cdot 2^{-\varepsilon(|t-t_2(n)|+l)} \rceil. \quad (119)$$

Let us estimate the sum

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{0 \leq l \leq \log r_t - 1} e_{\hat{k}_{t,l}}(\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t))$$

(notice that $\hat{P}_{\mathbf{T}_{t, \lfloor \log r_t \rfloor}} \stackrel{(95)}{=} 0$ since $\text{card } \mathbf{T}_{\hat{f},t, \lfloor \log r_t \rfloor} \stackrel{(113)}{=} 1$).

We set $T_{t,l} = \{\Omega_{\mathcal{A}'}\}_{\mathcal{A}' \in \mathbf{T}_{t,l}}$,

$$\hat{T}_{t,l} = \{E \cap E' : E \in T_{t,l}, E' \in T_{t,l+1}, \text{mes}(E \cap E') > 0\}.$$

By construction, for any function $f \in \hat{X}_p(\Omega)$ we have $\hat{P}_{\mathbf{T}_{t,l}} f - \hat{P}_{\mathbf{T}_{t,l+1}} f \stackrel{(98)}{\in} \mathcal{S}_{\hat{T}_{t,l}}(\Omega)$. Let $s'_{t,l} = \dim \mathcal{S}_{\hat{T}_{t,l}}(\Omega)$. From (110) and (111) it follows that there exists $C(\mathfrak{Z}_0) \geq 1$ such that

$$s'_{t,l} \leq \lceil C(\mathfrak{Z}_0) r_t \cdot 2^{-l} \rceil =: s''_{t,l}. \quad (120)$$

By Lemma 5, there exists an isomorphism $\overline{A}_{t,l} : \mathcal{S}_{\hat{T}_{t,l}}(\Omega) \rightarrow \mathbb{R}^{s'_{t,l}}$ such that

$$\|\overline{A}_{t,l}\|_{Y_{p,q,\hat{T}_{t,l}}(G_t) \rightarrow l_p^{s'_{t,l}}} \lesssim_{\mathfrak{Z}_0} 1, \quad \|\overline{A}_{t,l}^{-1}\|_{l_q^{s'_{t,l}} \rightarrow Y_q(G_t)} \lesssim_{\mathfrak{Z}_0} 1. \quad (121)$$

Hence, by (9) it suffices to estimate the sum

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{0 \leq l \leq \log r_t - 1} \|\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}}\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,\hat{T}_{t,l}}(\Omega)} e_{\hat{k}_{t,l}}(I_{s''_{t,l}} : l_p^{s''_{t,l}} \rightarrow l_q^{s''_{t,l}}) =: S.$$

Let us estimate $\|\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}}\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,\hat{T}_{t,l}}(\Omega)}$. Consider a function $f \in B\hat{X}_p(\Omega)$. Then

$$\begin{aligned} \|\hat{P}_{\mathbf{T}_{t,l}} f - \hat{P}_{\mathbf{T}_{t,l+1}} f\|_{p,q,\hat{T}_{t,l}} &= \left(\sum_{E \in \hat{T}_{t,l}} \|\hat{P}_{\mathbf{T}_{t,l}} f - \hat{P}_{\mathbf{T}_{t,l+1}} f\|_{Y_q(E)}^p \right)^{1/p} \stackrel{(111)}{\lesssim_{\mathfrak{Z}_0}} \\ &\left(\sum_{E' \in T_{t,l}} \|P_{t,m_t} f - Q_t f - \hat{P}_{\mathbf{T}_{t,l}} f\|_{Y_q(E')}^p + \sum_{E'' \in T_{t,l+1}} \|P_{t,m_t} f - Q_t f - \hat{P}_{\mathbf{T}_{t,l+1}} f\|_{Y_q(E'')}^p \right)^{1/p} \\ &\stackrel{(99),(100),(101)}{\lesssim_{\mathfrak{Z}_0}} 2^{-\lambda_* k_* t} u_*(2^{k_* t}); \end{aligned}$$

i.e.,

$$\|\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}}\|_{\hat{X}_p(\Omega) \rightarrow Y_{p,q,\hat{T}_{t,l}}(\Omega)} \lesssim_{\mathfrak{Z}_0} 2^{-\lambda_* k_* t} u_*(2^{k_* t}). \quad (122)$$

Since for $0 \leq l \leq \log r_t$ and small $\varepsilon > 0$

$$\frac{s''_{t,l}}{\hat{k}_{t,l}} \stackrel{(119),(120)}{\lesssim_{\mathfrak{Z}_0}} \frac{r_t \cdot 2^{-l}}{\lceil n \cdot 2^{-\varepsilon(|t-t_2(n)|+l)} \rceil} \stackrel{(105),(107)}{\lesssim_{\mathfrak{Z}_0}} 1,$$

by Theorem A we get

$$e_{\hat{k}_{t,l}}(I_{s''_{t,l}} : l_p^{s''_{t,l}} \rightarrow l_q^{s''_{t,l}}) \lesssim_{\mathfrak{Z}_0} (s''_{t,l})^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{-\frac{\hat{k}_{t,l}}{s''_{t,l}}}.$$

Moreover, there exists $m_* = m_*(\mathfrak{Z}_0)$ such that for $0 \leq \nu < m_*$ the sequence $\left\{ \frac{\hat{k}_{t,m_* l + \nu}}{s''_{t,m_* l + \nu}} \right\}_{0 \leq m_* l + \nu \leq \log r_t}$ increases not slower than some geometric progression. This together with (122) yields that

$$\begin{aligned} S &\stackrel{\mathfrak{Z}_0}{\lesssim} \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) \max_{0 \leq \nu < m_*} (s''_{t,\nu})^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{-\frac{\hat{k}_{t,\nu}}{s''_{t,\nu}}} \stackrel{\mathfrak{Z}_0}{\lesssim} \\ &\lesssim \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-\lambda_* k_* t} u_*(2^{k_* t}) (\hat{k}_{t,0})^{\frac{1}{q}-\frac{1}{p}} =: S'. \end{aligned}$$

If (21) holds, then

$$S' \stackrel{(119)}{\lesssim_{\mathfrak{Z}_0}} 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) (\hat{k}_{t_*(n),0})^{\frac{1}{q}-\frac{1}{p}} \stackrel{(47),(86)}{\lesssim_{\mathfrak{Z}_0}} n^{-\lambda_* \beta_* + \frac{1}{q}-\frac{1}{p}} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)).$$

Hence,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{0 \leq l \leq \log r_t - 1} e_{\hat{k}_{t,l}}(\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_* \beta_* + \frac{1}{q} - \frac{1}{p}} \varphi_*^{-\lambda_*}(n) u_*(n^{\beta_*} \varphi_*(n)). \quad (123)$$

Let (22) hold. Then for $\lambda_* > \frac{1}{p} - \frac{1}{q}$

$$S' \underset{\mathfrak{Z}_0}{\stackrel{(119)}{\lesssim}} 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) (\hat{k}_{t_*(n),0})^{\frac{1}{q} - \frac{1}{p}} \underset{\mathfrak{Z}_0}{\stackrel{(49),(86)}{\lesssim}} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_*} u_*(\log n).$$

Hence,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{0 \leq l \leq \log r_t - 1} e_{\hat{k}_{t,l}}(\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda_* + \frac{1}{p} - \frac{1}{q}} u_*(\log n). \quad (124)$$

Let $\lambda_* < \frac{1}{p} - \frac{1}{q}$. Then $\hat{k}_{t,0} \stackrel{(119)}{\geq} n \cdot 2^{-\varepsilon t_{**}(n)} \underset{\mathfrak{Z}_0}{\stackrel{(49)}{\lesssim}} n^{1-\varepsilon}$. Therefore, for sufficiently small $\varepsilon > 0$

$$\begin{aligned} S' &\underset{\mathfrak{Z}_0}{\lesssim} 2^{-\lambda_* k_* t_*(n)} u_*(2^{k_* t_*(n)}) (\hat{k}_{t_*(n),0})^{\frac{1}{q} - \frac{1}{p}} \underset{\mathfrak{Z}_0}{\stackrel{(49)}{\lesssim}} \\ &\lesssim n^{(\frac{1}{q} - \frac{1}{p})(1-\varepsilon)} (\log n)^{-\lambda_*} u_*(\log n) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_*} u_*(n); \end{aligned}$$

i.e.,

$$\sum_{t=t_*(n)}^{t_{**}(n)-1} \sum_{0 \leq l \leq \log r_t - 1} e_{\hat{k}_{t,l}}(\hat{P}_{\mathbf{T}_{t,l}} - \hat{P}_{\mathbf{T}_{t,l+1}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \underset{\mathfrak{Z}_0}{\lesssim} n^{-\lambda_*} u_*(n). \quad (125)$$

Step 7. Let $\mathcal{N}'_t = \{\mathbf{T}_{\hat{f},t,0}|_{\Gamma_t} : \hat{f} \in B\hat{X}_p(\Omega)\}$. Then $\mathcal{N}'_t \stackrel{(114)}{\subset} \mathcal{N}_{r_t-1}$, and

$$\log |\mathcal{N}'_t| \underset{\mathfrak{Z}_{0,\varepsilon}}{\stackrel{(86),(106)}{\lesssim}} n \cdot 2^{-\varepsilon |t-t_2(n)|} \quad \text{if (21) holds,} \quad (126)$$

$$\log |\mathcal{N}'_t| \underset{\mathfrak{Z}_{0,\varepsilon}}{\stackrel{(108)}{\lesssim}} \max\{n \cdot 2^{-\varepsilon |t-t_2(n)|}, 2^t\} \quad \text{if (22) holds.} \quad (127)$$

Let $\hat{k}_{t,l}$ be defined by (119). We set

$$\hat{k}_t = \sum_{0 \leq l \leq \log r_t - 1} (\hat{k}_{t,l} - 1) + 1 \underset{\mathfrak{Z}_{0,\varepsilon}}{\lesssim} n \cdot 2^{-\varepsilon |t-t_2(n)|}. \quad (128)$$

By Theorem C,

$$\begin{aligned}
& e_{\hat{k}_t + [\log |\mathcal{N}'_t|] + 1} (P_{t,m_t} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \leq \\
& \leq \sup_{\mathbf{T} \in \mathcal{N}'_t} e_{\hat{k}_t} (\hat{P}_{\mathbf{T}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) + \\
& + \sup_{f \in B\hat{X}_p(\Omega)} \inf_{\mathbf{T} \in \mathcal{N}'_t} \|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}}f\|_{Y_q(G_t)} \leq \\
& \leq \sup_{\hat{f} \in B\hat{X}_p(\Omega)} e_{\hat{k}_t} (\hat{P}_{\mathbf{T}_{\hat{f},t,0}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) + \\
& + \sup_{f \in B\hat{X}_p(\Omega)} \|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}_{f,t,0}}f\|_{Y_q(G_t)}.
\end{aligned} \tag{129}$$

Let $\tilde{k}_t = \tilde{k}_t(n) = \hat{k}_t + [\log |\mathcal{N}'_t|] + 1$. Then

$$\tilde{k}_t - 1 \stackrel{(126),(128)}{\underset{30,\varepsilon}{\lesssim}} n \cdot 2^{-\varepsilon|t-t_2(n)|} \quad \text{for} \quad (21),$$

$$\tilde{k}_t - 1 \stackrel{(127),(128)}{\underset{30,\varepsilon}{\lesssim}} \max\{n \cdot 2^{-\varepsilon|t-t_2(n)|}, 2^t\} \quad \text{for} \quad (22);$$

i.e., (93), (94) hold. Further,

$$\begin{aligned}
& \sum_{t=t_*(n)}^{t_{**}(n)-1} e_{\tilde{k}_t} (P_{t,m_t} - Q_t : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) \stackrel{(129)}{\leq} \\
& \leq \sum_{t=t_*(n)}^{t_{**}(n)-1} \sup_{\hat{f} \in B\hat{X}_p(\Omega)} e_{\hat{k}_t} (\hat{P}_{\mathbf{T}_{\hat{f},t,0}} : \hat{X}_p(\Omega) \rightarrow Y_q(G_t)) + \\
& + \sum_{t=t_*(n)}^{t_{**}(n)-1} \sup_{f \in B\hat{X}_p(\Omega)} \|P_{t,m_t}f - Q_tf - \hat{P}_{\mathbf{T}_{f,t,0}}f\|_{Y_q(G_t)}.
\end{aligned}$$

We apply (8) and (123), (124), (125) to estimate the first summand, and we use (115), (116), (117) and (118) to estimate the second summand.

This completes the proof. \square

The relations (8), (50), (51) and Lemmas 4, 6, 7, 8, 9 yield Theorems 4 and 5 for $p < q$.

Remark 6. Suppose that Assumptions 1 and 3 hold, and Assumption 2 is substituted by the following condition: for any $\xi \in \mathbf{V}(\mathcal{A})$ the set $F(\xi)$ is the atom of mes . Then the assertions of Theorems 4 and 5 hold with $\delta_* = +\infty$.

4 Estimates of entropy numbers of weighted Sobolev spaces

Let $\Omega \in \mathbf{FC}(a)$, and let $\Gamma \subset \partial\Omega$ be an h -set. Suppose that there exists $c_0 \geq c_*$ such that

$$\frac{h(t)}{h(s)} \leq c_0, \quad j \in \mathbb{N}, \quad t, s \in [2^{-j-1}, 2^{-j+1}] \quad (130)$$

(here c_* is the constant from Definition 4). In [39, 40] there were constructed a tree \mathcal{A} with vertex set $\{\eta_{j,i}\}_{j \geq j_{\min}, i \in \tilde{I}_j}$, a number $\bar{s} = \bar{s}(a, d) \in \mathbb{N}$ and a partition $\{\Omega[\eta_{j,i}]\}_{j \geq j_{\min}, i \in \tilde{I}_j}$ of the domain Ω with following properties:

1. $\eta_{j_{\min},1}$ is the minimal vertex in \mathcal{A} , and for any $j \geq j_{\min}$ the family of sets $\{\mathbf{V}_1^{\mathcal{A}}(\eta_{j,i})\}_{i \in \tilde{I}_j}$ form the partition of $\{\eta_{j+1,t}\}_{t \in \tilde{I}_{j+1}}$.
2. For any $j \geq j_{\min}$, $i \in \tilde{I}_j$ we have $\Omega[\eta_{j,i}] \in \mathbf{FC}(b_*)$ with $b_* = b_*(a, d) > 0$.
3. $\text{diam } \Omega[\eta_{j,i}] \underset{a,d,c_0}{\asymp} 2^{-\bar{s}j}$.
4. For any $x \in \Omega[\eta_{j,i}]$ we have $\text{dist}(x, \Gamma) \underset{a,d,c_0}{\asymp} 2^{-\bar{s}j}$.
5. For any $j \geq j_{\min}$, $i \in \tilde{I}_j$, $j' \geq j$ we have $\text{card } \mathbf{V}_{j'-j}^{\mathcal{A}}(\eta_{j,i}) \underset{a,d,c_0}{\lesssim} \frac{h(2^{-\bar{s}j})}{h(2^{-\bar{s}j'})}$; in particular,

$$\text{card } \tilde{I}_j \underset{a,d,c_0}{\lesssim} \frac{h(2^{-\bar{s}j_{\min}})}{h(2^{-\bar{s}j})}. \quad (131)$$

Suppose that conditions of Theorem 1, 2 or 3 hold (then we have (130) with $c_0 = c_0(\mathfrak{Z})$). We define weight functions $u, w : \mathbf{V}(\mathcal{A}) \rightarrow (0, \infty)$ as follows:

$$u(\eta_{j,i}) = u_j = g(2^{-\bar{s}j}) \cdot 2^{-(r-\frac{d}{p})\bar{s}j}, \quad w(\eta_{j,i}) = w_j = v(2^{-\bar{s}j}) \cdot 2^{-\frac{d}{q}\bar{s}j}. \quad (132)$$

For each subtree $\mathcal{D} \subset \mathcal{A}$ we denote $\Omega[\mathcal{D}] = \cup_{\xi \in \mathbf{V}(\mathcal{D})} \Omega[\xi]$. It was proved in [40, 44, 45] that for any $j_0 \geq j_{\min}$, $i_0 \in \tilde{I}_{j_0}$ and for any vertex η_{j_0,i_0} there exists a linear continuous operator $P_{\eta_{j_0,i_0}} : L_{q,v}(\Omega) \rightarrow \mathcal{P}_{r-1}(\Omega)$ such that for any subtree \mathcal{D} with minimal vertex η_{j_0,i_0} and for any function $f \in W_{p,g}^r(\Omega)$

$$\|f - P_{\eta_{j_0,i_0}} f\|_{L_{q,v}(\Omega[\mathcal{D}])} \lesssim_3 C(j_0) \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}; \quad (133)$$

here $C(j_0)$ is defined as follows.

1. Let $\beta_v < \frac{d-\theta}{q}$. Then (see [40])

$$C(j_0) = \sup_{j \geq j_0} u_j w_j \quad \text{for } p \leq q,$$

$$C(j_0) = \left(\sum_{j \geq j_0} (u_j w_j)^{\frac{pq}{p-q}} \frac{h(2^{-\bar{s}j_0})}{h(2^{-\bar{s}j})} \right)^{\frac{1}{q} - \frac{1}{p}} \quad \text{for } p > q.$$

2. Let $\theta > 0$, $\beta_v = \frac{d-\theta}{q}$. Then (see [45])

$$\begin{aligned} C(j_0) &= 2^{-(\delta-\beta)\bar{s}j_0} (\bar{s}j_0)^{-\alpha+\frac{1}{q}} \rho(\bar{s}j_0), \quad \text{if } p < q \\ \text{or } p &\geq q, \quad \beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right), \end{aligned} \quad (134)$$

$$C(j_0) = 2^{-\theta(\frac{1}{q}-\frac{1}{p})\bar{s}j_0} (\bar{s}j_0)^{-\alpha+1+\frac{1}{q}-\frac{1}{p}} \rho(\bar{s}j_0) \quad \text{for } p \geq q, \quad \beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right). \quad (135)$$

The magnitude $C(j_0)$ in Case 1 is estimated as follows. If $\beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$, then

$$C(j_0) \underset{3}{\lesssim} 2^{(\beta-\delta)\bar{s}j_0} (\bar{s}j_0)^{-\alpha} \rho(\bar{s}j_0), \quad (136)$$

if $\beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$ and $\alpha > (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+$, then

$$C(j_0) \underset{3}{\lesssim} 2^{-\theta(\frac{1}{q}-\frac{1}{p})_+ \bar{s}j_0} j_0^{-\alpha+(\frac{1}{q}-\frac{1}{p})_+} \rho(j_0) \quad (137)$$

(these estimates are proved in [40]). Suppose that conditions of assertion 2 of Theorem 3 hold: i.e., $\theta = 0$,

$$\beta - \delta = 0, \quad \alpha = (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+, \quad \lambda > (1 - \nu) \left(\frac{1}{q} - \frac{1}{p} \right)_+. \quad (138)$$

Then

$$C(j_0) \stackrel{(4),(132)}{\underset{3}{\lesssim}} (\log(\bar{s}j_0))^{-\lambda}, \quad p \leq q, \quad (139)$$

$$\begin{aligned} C(j_0) &\stackrel{(2),(4),(132),(138)}{\underset{3}{\lesssim}} (\bar{s}j_0)^{\gamma(\frac{1}{q}-\frac{1}{p})} [\log(\bar{s}j_0)]^{\nu(\frac{1}{q}-\frac{1}{p})} \left(\sum_{j \geq j_0} (\bar{s}j)^{-1} (\log(\bar{s}j))^{-\lambda\frac{pq}{p-q}-\nu} \right)^{\frac{1}{q}-\frac{1}{p}} \underset{3}{\lesssim} \\ &\underset{3}{\lesssim} (\bar{s}j_0)^{\gamma(\frac{1}{q}-\frac{1}{p})} [\log(\bar{s}j_0)]^{-\lambda+\frac{1}{q}-\frac{1}{p}}, \quad p > q. \end{aligned} \quad (140)$$

In proofs of lower estimates of entropy numbers we use the following assertions.

Lemma 11. Let $\Omega \subset \mathbb{R}^d$ be a domain, let $W_{p,g}^r(\Omega) \subset L_{q,v}(\Omega)$, let $G_1, \dots, G_m \subset \Omega$ be pairwise non-overlapping sets, and let $\psi_1, \dots, \psi_m \in W_{p,g}^r(\Omega)$, $\left\| \frac{\nabla^r \psi_j}{g} \right\|_{L_p(\Omega)} = 1$, $\text{supp } \psi_j \subset G_j$,

$$\|\psi_j\|_{L_{q,v}(G_j)} \geq M, \quad 1 \leq j \leq m. \quad (141)$$

Let $X = \text{span} \{\psi_j\}_{j=1}^m$ be equipped with norm $\|f\|_X = \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)}$, and let $\text{id} : X \rightarrow L_{q,v}(\Omega)$ be the embedding operator. Then for any $n \in \{0, \dots, m\}$

$$e_n(\text{id} : X \rightarrow L_{q,v}(\Omega)) \geq M \cdot e_n(I_m : l_p^m \rightarrow l_q^m).$$

In particular, if $X \subset \hat{\mathcal{W}}_{p,g}^r(\Omega)$, then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \geq M \cdot e_n(I_m : l_p^m \rightarrow l_q^m).$$

This lemma is proved similarly as the lower estimate of n -widths in [47]. Similarly from Theorem B we obtain

Corollary 1. Let $\Omega \subset \mathbb{R}^d$ be a domain, let $W_{p,g}^r(\Omega) \subset L_{q,v}(\Omega)$, let $G_j \subset \Omega$, $j \in \mathbb{N}$, be pairwise non-overlapping sets, and let $\psi_j \in W_{p,g}^r(\Omega)$, $\left\| \frac{\nabla^r \psi_j}{g} \right\|_{L_p(\Omega)} = 1$, $\text{supp } \psi_j \subset G_j$,

$$\|\psi_j\|_{L_{q,v}(G_j)} \geq M_j, \quad j \in \mathbb{N}. \quad (142)$$

Let $X = \text{span} \{\psi_j\}_{j=1}^\infty \cap \text{span } W_{p,g}^r(\Omega)$ be equipped with norm $\|f\|_X = \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)}^{\frac{1}{q} - \frac{1}{p}}$, and let $\text{id} : X \rightarrow L_{q,v}(\Omega)$ be the embedding operator. Denote $\omega_n = \left(\sum_{j=n}^\infty M_j^{\frac{pq}{p-q}} \right)^{\frac{1}{q} - \frac{1}{p}}$.

Suppose that there exists $C \geq 1$ such that $\omega_n \leq C\omega_{2n}$ for any $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ we have

$$e_n(\text{id} : X \rightarrow L_{q,v}(\Omega)) \underset{p,q,C}{\gtrsim} \omega_n.$$

In particular, if $X \subset \hat{\mathcal{W}}_{p,g}^r(\Omega)$, then

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{p,q,C}{\gtrsim} \omega_n.$$

Proof of Theorems 1, 2, 3. *The upper estimate.* We set $\hat{\Theta} = \{\Omega[\eta_{j,i}]\}_{j \geq j_{\min}, i \in \bar{I}_j}$, $\hat{F}(\eta_{j,i}) = \Omega[\eta_{j,i}]$, $X_p(\Omega) = \text{span } W_{p,g}^r(\Omega)$, $\hat{X}_p(\Omega) = \hat{\mathcal{W}}_{p,g}^r(\Omega)$, $Y_q(\Omega) = L_{q,v}(\Omega)$, $\mathcal{P}(\Omega) = \mathcal{P}_{r-1}(\Omega)$.

Similarly as in [42, p. 49] we can prove that Assumption 2 holds with $\delta_* = \frac{\delta}{d}$ and $\tilde{w}_*(\eta_{j,i}) \lesssim u_j w_j$.

Consider the following cases.

1. Suppose that one of the following conditions holds:

- $\beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$,
- $\beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+, p \geq q$,
- $\theta = 0, \beta - \delta = 0, \alpha > (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+$.

Then the partition $\{\mathcal{A}_{t,i}\}_{t \geq t_0, i \in \hat{J}_t}$ is constructed similarly as in [42, p. 49–50], and by properties 1–5 of the tree \mathcal{A} and (133), (134), (135), (136), (137) we get that Assumptions 1 and 3 hold. Here λ_* , γ_* and ψ_* are the same as in [42] (see Cases 1, 3, 4, and Case 2 for $p \geq q$; in Cases 1 and 4 we take $\Lambda(x) = |\log x|^\gamma \tau(|\log x|)$). If $\beta_v < \frac{d-\theta}{q}$, then the function u_* is the same as in [42] (in Cases 1 and 4 we take $\Psi(x) = |\log x|^{-\alpha} \rho(|\log x|)$). If $\beta_v = \frac{d-\theta}{q}$, then $u_*(y) = (\log y)^{-\alpha + \frac{1}{q}} \rho(\log y)$ for $p < q$ or $\beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$, and $u_*(y) = (\log y)^{-\alpha + 1 + \frac{1}{q} - \frac{1}{p}} \rho(\log y)$ for $\beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$ and $p \geq q$ (recall that for $\beta_v = \frac{d-\theta}{q}$ we consider only the case $\theta > 0$).

Applying Theorem 4 and Lemma 2, we get the upper estimate in assertions 1 and 2a of Theorem 1, in Theorem 2 and in assertion 1 of Theorem 3.

2. Let $\theta > 0, \beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+, p < q$. Denote by Γ_t the maximal subgraph in \mathcal{A} on the vertex set

$$\{\eta_{j,l} : 2^{t-1} < \bar{s}j \leq 2^t, \quad l \in \tilde{I}_j\},$$

and by $\{\mathcal{A}_{t,i}\}_{i \in \hat{J}_t}$ we denote the set of connected components of Γ_t . We set $t_0 = \min\{t \in \mathbb{Z}_+ : \mathbf{V}(\Gamma_t) \neq \emptyset\}$. By (2) and (131), $\text{card } \mathbf{V}(\Gamma_t) \underset{30}{\lesssim} 2^{\theta \cdot 2^t} 2^{-\gamma t} \tau^{-1}(2^t)$.

This together with (133), (134) and (137) implies that Assumptions 1 and 3 hold with $\lambda_* = -\alpha_0$ (see assertion 2b of Theorem 1), $u_*(y) = \rho(y)$, $\gamma_* = \theta$, $\psi_*(y) = (\log y)^{-\gamma} \tau^{-1}(\log y)$ in (22). Applying Theorem 5, we get the upper estimate in assertion 2b of Theorem 1.

3. Let $\theta = 0, \beta - \delta = 0, \alpha = (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+$.

- (a) Suppose that $p \geq q$. We define the partition $\{\mathcal{A}_{t,i}\}_{t \geq t_0, i \in \hat{J}_t}$ similarly as in [42, p. 50] (see Case 3). Then Assumptions 1 and 3 hold with $\gamma_* = 1 - \gamma$, $\lambda_* = \alpha + \frac{1}{p} - \frac{1}{q} = -\gamma \left(\frac{1}{q} - \frac{1}{p} \right)$, $\mu_* = \alpha = (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)$, $\psi_*(x) = (\log x)^{-\nu}$, $u_*(x) = (\log x)^{-\lambda + \frac{1}{q} - \frac{1}{p}}$ (see (139), (140)). Hence, $\beta_* = \frac{1}{1-\gamma}$, $\varphi_*(x) = (\log x)^{\frac{\nu}{1-\gamma}}$ (see Lemma 2). In addition, $\text{card } \hat{J}_t \underset{3}{\lesssim}^{(131)}$

$2^{-\gamma t} t^{-\nu}$. The relations (23) and (25) follow from the conditions $\alpha = (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)$ and $\lambda > (1 - \nu) \left(\frac{1}{q} - \frac{1}{p} \right)$. Let us check that (24) holds. Indeed, $2^{-\lambda_* k_* t} u_*(2^{k_* t}) = 2^{\gamma \left(\frac{1}{q} - \frac{1}{p} \right) t} t^{-\lambda + \frac{1}{q} - \frac{1}{p}}$. Since the function h is non-decreasing, we have $\gamma \leq 0$; moreover, $\nu \leq 0$ for $\gamma = 0$. If $\gamma < 0$ and $p > q$, then (24) follows from the inequality $\gamma \left(\frac{1}{q} - \frac{1}{p} \right) < 0$. If $\gamma = 0$ and $p > q$, then (24) follows from the inequality $-\lambda + \frac{1}{q} - \frac{1}{p} < \nu \left(\frac{1}{q} - \frac{1}{p} \right) \leq 0$. If $p = q$, then the assertion follows from the inequality $\lambda > 0$. Notice that for $p = q$ we have $\lambda_* = \mu_*$. Since $\mu_* \beta_* = \frac{1}{q} - \frac{1}{p} < \frac{\delta}{d} = \delta_*$, we get by Theorem 4 that

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{3}{\lesssim} u_*(n^{\beta_*} \varphi_*(n)) \varphi_*^{-\mu_*}(n) \underset{3}{\asymp} (\log n)^{-\lambda + \frac{1}{q} - \frac{1}{p} - \nu \left(\frac{1}{q} - \frac{1}{p} \right)}.$$

Thus, we obtain the upper estimate in assertion 2a of Theorem 3.

- (b) Let $p < q$. We set $\mathbf{V}(\Gamma_t) = \{\eta_{j,i} : 2^{2^{t-1}} < j \leq 2^{2^t}\}$ and denote by $\{\mathcal{A}_{t,i}\}_{i \in \hat{J}_t}$ the set of connected components of the graph Γ_t . Then

$$\text{card } \mathbf{V}(\Gamma_t) \underset{3}{\lesssim} \sum_{2^{2^{t-1}+1} \leq \bar{s}j \leq 2^{2^t}} (\bar{s}j)^{-\gamma} (\log(\bar{s}j))^{-\nu} \underset{3}{\lesssim} 2^{(1-\gamma)2^t} 2^{-\nu t} =: \bar{\nu}_t$$

(i.e., (22) holds). This together with (139) implies that Assumptions 1 and 3 hold with $\lambda_* = \mu_* = \lambda$, $u_* \equiv 1$. Applying Theorem 5, we get the upper estimate in assertion 2b of Theorem 3.

The lower estimate. Similarly as in [42, p. 50] we can prove that

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \underset{3_*}{\gtrsim} e_n(I : \hat{\mathcal{W}}_p^r([0, 1]^d) \rightarrow L_q([0, 1]^d)) \underset{p,q,r,d}{\asymp} n^{-\frac{\delta}{d} + \frac{1}{q} - \frac{1}{p}}.$$

This gives the desired lower estimates in Theorem 2, in assertion 1 of Theorem 1 for $\frac{\delta}{d} < \frac{\delta - \beta}{\theta}$ and in assertion 1 of Theorem 3 for $\frac{\delta}{d} < \frac{\alpha}{1 - \gamma}$.

In other cases we apply Lemma 11 or Corollary 1.

In [42, p. 50], [40] and [45] the number $k_{**} = k_{**}(\mathfrak{Z}_*) \in \mathbb{N}$ is defined and the functions $\{\psi_{t,j}\}_{j \in J_t} \in C^\infty(\mathbb{R}^d)$ are constructed with the following properties:

$$\text{card } J_t \underset{3_*}{\gtrsim} 2^{\theta k_{**} t} (k_{**} t)^{-\gamma} \tau^{-1} (k_{**} t), \quad (143)$$

$\left\| \frac{\nabla^r \psi_{t,j}}{g} \right\|_{L_p(\Omega)} = 1$. Moreover, $\|\psi_{t,j}\|_{L_{q,v}(\Omega)}$ is estimated from below as follows.

1. If $\beta_v < \frac{d - \theta}{q}$, then

$$\|\psi_{t,j}\|_{L_{q,v}(\Omega)} \underset{3_*}{\gtrsim} 2^{k_{**} t (\beta - \delta)} (k_{**} t)^{-\alpha} \rho(k_{**} t). \quad (144)$$

2. Let $\theta > 0$, $\beta_v = \frac{d-\theta}{q}$; in addition, we suppose that $p < q$ or $p \geq q$, $\beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$. Then

$$\|\psi_{t,j}\|_{L_{q,v}(\Omega)} \gtrsim_{\mathfrak{Z}_*} 2^{k_{**}t(\beta-\delta)} (k_{**}t)^{-\alpha+\frac{1}{q}} \rho(k_{**}t). \quad (145)$$

3. If $\theta > 0$, $p \geq q$, $\beta_v = \frac{d-\theta}{q}$ and $\beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)$, then

$$\|\psi_{t,j}\|_{L_{q,v}(\Omega)} \gtrsim_{\mathfrak{Z}_*} 2^{-\theta \left(\frac{1}{q} - \frac{1}{p} \right) k_{**}t} (k_{**}t)^{-\alpha+\frac{1}{q}+1-\frac{1}{p}} \rho(k_{**}t). \quad (146)$$

In addition, in Case 1 the supports of $\psi_{t,j}$ do not overlap pairwise for different (t, j) ; in Cases 2, 3 for any t the supports of $\psi_{t,j}$ do not overlap pairwise for different j .

Moreover, it follows from the construction of functions $\psi_{t,i}$ that for any $x \in \text{supp } \psi_{t,j}$

$$\text{dist}(x, \Gamma) \lesssim_{\mathfrak{Z}_*} 2^{-k_{**}t}.$$

Hence, by Remark 1, there exists $\hat{t} = \hat{t}(\mathfrak{Z}_*) \in \mathbb{N}$ such that for $t \geq \hat{t}$ we have $P\psi_{t,i} = 0$ and $\psi_{t,i} \in \hat{W}_{p,g}^r(\Omega)$.

1. Suppose that $\theta > 0$.

- (a) Let $\beta - \delta < -\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$. Let α_0 be as defined in assertion 1 of Theorem 1. We take t_n such that

$$n \leq 2^{\theta k_{**}t_n} (k_{**}t_n)^{-\gamma} \tau^{-1} (k_{**}t_n) \lesssim_{\mathfrak{Z}_*} n. \quad (147)$$

Then

$$2^{k_{**}t_n} \gtrsim_{\mathfrak{Z}_*} n^{\frac{1}{\theta}} (\log n)^{\frac{\gamma}{\theta}} \tau^{\frac{1}{\theta}} (\log n) \quad (148)$$

(see Lemma 2) and

$$k_{**}t_n \gtrsim_{\mathfrak{Z}_*} \log n. \quad (149)$$

This together with Theorem A and Lemma 11 yields that

$$\begin{aligned} e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) &\stackrel{(144),(145)}{\gtrsim_{\mathfrak{Z}_*}} n^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{k_{**}t_n(\beta-\delta)} (k_{**}t_n)^{-\alpha_0} \rho(k_{**}t_n) \asymp_{\mathfrak{Z}_*} \\ &\asymp n^{\frac{\beta-\delta}{\theta}+\frac{1}{q}-\frac{1}{p}} (\log n)^{\frac{(\beta-\delta)\gamma}{\theta}-\alpha_0} \rho(\log n) \tau^{\frac{\beta-\delta}{\theta}} (\log n). \end{aligned}$$

This implies the lower estimate in assertion 1 of Theorem 1.

(b) Let $\beta - \delta = -\theta \left(\frac{1}{q} - \frac{1}{p} \right)$, $p \geq q$. We take t_n such that (147) holds.

If $\beta_v < \frac{d-\theta}{q}$ and $p = q$, then Theorem A and Lemma 11 yield that

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \stackrel{(144),(149)}{\underset{3_*}{\gtrsim}} (\log n)^{-\alpha} \rho(\log n).$$

Let $\beta_v < \frac{d-\theta}{q}$, $p > q$. We apply Corollary 1 and get that there exists such $m_0 = m_0(3_*)$ that

$$\begin{aligned} e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) &\stackrel{(144)}{\underset{3_*}{\gtrsim}} \\ &\gtrsim \left(\sum_{t \geq t_n + m_0} \sum_{i \in J_t} 2^{-k_{**}t\theta} (k_{**}t)^{\frac{pq}{p-q}\alpha} \rho^{\frac{pq}{p-q}}(k_{**}t) \right)^{\frac{1}{q} - \frac{1}{p}} \stackrel{(143)}{\underset{3_*}{\gtrsim}} \\ &\gtrsim \left(\sum_{t \geq t_n + m_0} (k_{**}t)^{-\gamma} \tau^{-1}(k_{**}t) (k_{**}t)^{\frac{pq}{p-q}\alpha} \rho^{\frac{pq}{p-q}}(k_{**}t) \right)^{\frac{1}{q} - \frac{1}{p}} \underset{3_*}{\gtrsim} \\ &\gtrsim (k_{**}t_n)^{-\alpha + (1-\gamma)(\frac{1}{q} - \frac{1}{p})} \rho(k_{**}t_n) \tau^{\frac{1}{p} - \frac{1}{q}}(k_{**}t_n) \stackrel{(149)}{\underset{3_*}{\gtrsim}} \\ &\gtrsim (\log n)^{-\alpha + (1-\gamma)(\frac{1}{q} - \frac{1}{p})} \rho(\log n) \tau^{\frac{1}{p} - \frac{1}{q}}(\log n). \end{aligned}$$

Let $\beta_v = \frac{d-\theta}{q}$. Then Theorem A and Lemma 11 imply that

$$\begin{aligned} e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) &\stackrel{(146)}{\underset{3_*}{\gtrsim}} 2^{-\theta(\frac{1}{q} - \frac{1}{p})k_{**}t_n} (k_{**}t_n)^{-\alpha + \frac{1}{q} + 1 - \frac{1}{p}} \rho(k_{**}t_n) n^{\frac{1}{q} - \frac{1}{p}} \stackrel{(148),(149)}{\underset{3_*}{\gtrsim}} \\ &\gtrsim (\log n)^{-\gamma(\frac{1}{q} - \frac{1}{p}) - \alpha + \frac{1}{q} + 1 - \frac{1}{p}} \rho(\log n) \tau^{\frac{1}{p} - \frac{1}{q}}(\log n). \end{aligned}$$

Thus, we obtain the desired estimate in assertion 2a of Theorem 1.

(c) Let $\beta - \delta = 0$, $p < q$. We take t_n such that

$$n^2 \leq 2^{\theta k_{**}t_n} (k_{**}t_n)^{-\gamma} \tau^{-1}(k_{**}t_n) \underset{3_*}{\lesssim} n^2. \quad (150)$$

Then $k_{**}t_n \underset{3_*}{\asymp} \log n$. Applying Theorem A and Lemma 11, we obtain that

$$\begin{aligned} e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) &\stackrel{(143),(144),(145),(150)}{\underset{3_*}{\gtrsim}} \\ &\gtrsim (k_{**}t_n)^{-\alpha_0} \rho(k_{**}t_n) n^{\frac{1}{q} - \frac{1}{p}} \log^{\frac{1}{p} - \frac{1}{q}} \left(1 + \frac{n^2}{n} \right) \underset{3_*}{\asymp} n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\alpha_0 + \frac{1}{p} - \frac{1}{q}} \rho(\log n). \end{aligned}$$

Now we take t_n such that

$$2^n \leq 2^{\theta k_{**} t_n} (k_{**} t_n)^{-\gamma} \tau^{-1}(k_{**} t_n) \lesssim_{3_*} 2^n.$$

Then $k_{**} t_n \asymp_{3_*} n$. Applying Theorem A and Lemma 11, we obtain that

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \stackrel{(144),(145)}{\underset{3_*}{\gtrsim}} (k_{**} t_n)^{-\alpha_0} \rho(k_{**} t_n) \asymp_{3_*} n^{-\alpha_0} \rho(n).$$

Thus, we get desired estimates in assertion 2b of Theorem 1.

2. Let $\theta = 0$ and conditions of Theorem 3 hold. Then

$$\|\psi_{t,j}\|_{L_{q,v}(\Omega)} \stackrel{(144)}{\underset{3_*}{\gtrsim}} (k_{**} t)^{-\alpha} \log^{-\lambda}(k_{**} t), \quad \text{card } J_t \stackrel{(143)}{\underset{3_*}{\gtrsim}} (k_{**} t)^{-\gamma} \log^{-\nu}(k_{**} t).$$

Hence, for $2^m \leq k_{**} t < 2^{m+1}$ and sufficiently large $m \in \mathbb{N}$

$$\|\psi_{t,j}\|_{L_{q,v}(\Omega)} \underset{3_*}{\gtrsim} 2^{-\alpha m} m^{-\lambda}, \quad \text{card } (\cup_{2^m \leq k_{**} t < 2^{m+1}} J_t) \underset{3_*}{\gtrsim} 2^{m(1-\gamma)} m^{-\nu}. \quad (151)$$

(a) Let $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ > 0$. We take $m_n \in \mathbb{N}$ such that

$$n \leq 2^{m_n(1-\gamma)} m_n^{-\nu} \lesssim_{3_*} n. \quad (152)$$

Then $2^{m_n} \asymp_{3_*} n^{\frac{1}{1-\gamma}} (\log n)^{\frac{\nu}{1-\gamma}}$ (see Lemma 2), $m_n \asymp_{3_*} \log n$. Applying Theorem A and Lemma 11, we get that

$$\begin{aligned} e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) &\stackrel{(151)}{\underset{3_*}{\gtrsim}} 2^{-\alpha m_n} m_n^{-\lambda} n^{\frac{1}{q} - \frac{1}{p}} \underset{3_*}{\asymp} \\ &\asymp n^{-\frac{\alpha}{1-\gamma} + \frac{1}{q} - \frac{1}{p}} (\log n)^{-\frac{\alpha\nu}{1-\gamma} - \lambda}. \end{aligned}$$

Thus, we obtain the desired estimate in assertion 1 of Theorem 3.

(b) Let $p \geq q$, $\alpha = (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)$. We define m_n by (152). For $p = q$ we have $\alpha = 0$ and

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \stackrel{(151),(152)}{\underset{3_*}{\gtrsim}} (\log n)^{-\lambda}.$$

If $p > q$, then we apply Corollary 1 and get that there exists $\hat{m} = \hat{m}(3_*)$ such that

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \stackrel{(151)}{\underset{3_*}{\gtrsim}}$$

$$\begin{aligned}
&\gtrsim \left(\sum_{m=m_n+\hat{m}}^{\infty} 2^{m(1-\gamma)} m^{-\nu} \cdot 2^{-m\alpha \frac{pq}{p-q}} m^{-\lambda \frac{pq}{p-q}} \right)^{\frac{1}{q}-\frac{1}{p}} \underset{\mathfrak{Z}_*}{\asymp} \\
&\asymp m_n^{-\lambda+(1-\nu)(\frac{1}{q}-\frac{1}{p})} \underset{\mathfrak{Z}_*}{\asymp} (\log n)^{-\lambda+(1-\nu)(\frac{1}{q}-\frac{1}{p})}.
\end{aligned}$$

Thus, we get the desired estimate in assertion 2a of Theorem 3.

(c) Let $p < q$, $\alpha = 0$. First we take m_n such that

$$2^{(1-\gamma)m_n} m_n^{-\nu} \underset{\mathfrak{Z}_*}{\asymp} n^2. \quad (153)$$

Then $m_n \underset{\mathfrak{Z}_*}{\asymp} \log n$. Applying Theorem A and Lemma 11, we get

$$\begin{aligned}
&e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \stackrel{(151),(153)}{\underset{\mathfrak{Z}_*}{\gtrsim}} \\
&\gtrsim m_n^{-\lambda} n^{\frac{1}{q}-\frac{1}{p}} \log^{\frac{1}{p}-\frac{1}{q}} \left(1 + \frac{n^2}{n} \right) \underset{\mathfrak{Z}_*}{\asymp} n^{\frac{1}{q}-\frac{1}{p}} (\log n)^{-\lambda+\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

Now we take m_n such that $2^{(1-\gamma)m_n} m_n^{-\nu} \underset{\mathfrak{Z}_*}{\asymp} 2^n$. Then $m_n \underset{\mathfrak{Z}_*}{\asymp} n$. Applying Theorem A and Lemma 11, we get

$$e_n(I : \hat{\mathcal{W}}_{p,g}^r(\Omega) \rightarrow L_{q,v}(\Omega)) \stackrel{(151)}{\underset{\mathfrak{Z}_*}{\gtrsim}} m_n^{-\lambda} \underset{\mathfrak{Z}_*}{\asymp} n^{-\lambda}.$$

Thus, we obtain the desired estimate in assertion 2b of Theorem 3.

This completes the proof of Theorems 1, 2, 3. \square

5 Estimates of entropy numbers of weighted summation operators on a tree

Let \mathcal{A} be a tree, and let $f : \mathbf{V}(\mathcal{A}) \rightarrow \mathbb{R}$. We set

$$\|f\|_{l_p(\mathcal{A})} = \left(\sum_{\xi \in \mathbf{V}(\mathcal{A})} |f(\xi)|^p \right)^{1/p}, \quad \text{if } 1 \leq p < \infty, \quad \|f\|_{l_\infty(\mathcal{A})} = \sup_{\xi \in \mathbf{V}(\mathcal{A})} |f(\xi)|.$$

Denote by $l_p(\mathcal{A})$ the space of functions $f : \mathbf{V}(\mathcal{A}) \rightarrow \mathbb{R}$ with finite norm $\|f\|_{l_p(\mathcal{A})}$.

Let $u, w : \mathbf{V}(\mathcal{A}) \rightarrow [0, \infty)$ be weight functions.

Define the summation operator $S_{u,w,\mathcal{A}}$ by

$$S_{u,w,\mathcal{A}}f(\xi) = w(\xi) \sum_{\xi' \leq \xi} u(\xi') f(\xi'), \quad \xi \in \mathbf{V}(\mathcal{A}), \quad f : \mathbf{V}(\mathcal{A}) \rightarrow \mathbb{R}.$$

In papers of Lifshits and Linde [26–28] estimates for entropy numbers of the operator $S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_\infty(\mathcal{A})$ or its dual were obtained under some conditions on u, w . Here we obtain order estimates for entropy numbers of the operator $S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})$ for $1 < p \leq \infty, 1 \leq q < \infty$.

Let $\mathbf{V}(\mathcal{A}) = \{\eta_{j,i} : j \in \mathbb{Z}_+, i \in I_j\}$. In addition, we suppose that the family of sets $\{\mathbf{V}_1^{\mathcal{A}}(\eta_{j,i})\}_{i \in I_j}$ forms the partition of the set $\{\eta_{j+1,t}\}_{t \in I_{j+1}}$. Suppose that for some $c_* \geq 1, m_* \in \mathbb{N}$

$$c_*^{-1} \frac{h(2^{-m_*j})}{h(2^{-m_*(j+l)})} \leq \mathbf{V}_l^{\mathcal{A}}(\eta_{j,i}) \leq c_* \frac{h(2^{-m_*j})}{h(2^{-m_*(j+l)})}, \quad j, l \in \mathbb{Z}_+.$$

Here the function h is defined by (2), (3) in some neighborhood of zero. Let $u, w : \mathbf{V}(\mathcal{A}) \rightarrow (0, \infty)$, $u(\eta_{j,i}) = u_j, w(\eta_{j,i}) = w_j, j \in \mathbb{Z}_+, i \in I_j$,

$$u_j = 2^{-\kappa_u m_* j} (m_* j + 1)^{-\alpha_u} \rho_u(m_* j + 1), \quad w_j = 2^{-\kappa_w m_* j} (m_* j + 1)^{-\alpha_w} \rho_w(m_* j + 1),$$

where $\rho_u, \rho_w : (0, \infty) \rightarrow (0, \infty)$ are absolutely continuous functions, $\lim_{y \rightarrow \infty} \frac{y \rho'_u(y)}{\rho_u(y)} = \lim_{y \rightarrow \infty} \frac{y \rho'_w(y)}{\rho_w(y)} = 0$. Moreover, we suppose that $1 < p \leq \infty, 1 \leq q < \infty$,

$$\kappa_w > \frac{\theta}{q} \quad \text{or} \quad \kappa_w = \frac{\theta}{q}, \quad \alpha_w > \frac{1-\gamma}{q}. \quad (154)$$

We set $\kappa = \kappa_u + \kappa_w, \alpha = \alpha_u + \alpha_w, \rho(y) = \rho_u(y) \rho_w(y), \mathfrak{Z} = (p, q, u, w, h, m_*, c_*)$.

Let \mathcal{D} be a subtree in \mathcal{A} . Denote by $\mathfrak{S}_{\mathcal{D},u,w}^{p,q}$ the operator norm of $S_{u,w,\mathcal{D}} : l_p(\mathcal{D}) \rightarrow l_q(\mathcal{D})$. Applying the results of [40], [46], we get that for $j \geq 2$ and for any $i \in I_j$ we have $\mathfrak{S}_{\mathcal{A}_{\eta_{j,i}},u,w}^{p,q} \asymp_3 C(j)$, where $C(j)$ is defined as follows.

1. Let $\kappa_w > \frac{\theta}{q}, \kappa > \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$. Then $C(j) = 2^{-\kappa m_* j} (m_* j)^{-\alpha} \rho(m_* j)$.

2. Let $\kappa_w > \frac{\theta}{q}, \kappa = \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+, \alpha > (1-\gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+$. Then

$$C(j) = 2^{-\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+ m_* j} (m_* j)^{-\alpha + \left(\frac{1}{q} - \frac{1}{p} \right)_+} \rho(m_* j).$$

3. Let $\theta > 0, \kappa_w = \frac{\theta}{q}$. Suppose that either $\kappa > \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$ or $\kappa = \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+, \alpha > \frac{1}{q}, p < q$. Then $C(j) = 2^{-\kappa m_* j} (m_* j)^{-\alpha + \frac{1}{q}} \rho(m_* j)$.

4. Let $\theta > 0, \kappa_w = \frac{\theta}{q}, \kappa = \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+, p \geq q, \alpha > 1 + (1-\gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+$. Then $C(j) = 2^{-\theta \left(\frac{1}{q} - \frac{1}{p} \right)_+ m_* j} (m_* j)^{-\alpha + 1 + \frac{1}{q} - \frac{1}{p}} \rho(m_* j)$.

Applying 6, we get results similar to Theorems 1, 3.

Theorem 6. *Let $\theta > 0$.*

1. Suppose that $\kappa > \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$. We set $\alpha_0 = \alpha$ for $\kappa_w > \frac{\theta}{q}$ and $\alpha_0 = \alpha - \frac{1}{q}$ for $\kappa_w = \frac{\theta}{q}$. Then

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 n^{-\frac{\kappa}{\theta} + \frac{1}{q} - \frac{1}{p}} (\log n)^{-\alpha_0 - \frac{\kappa\gamma}{\theta}} \rho(\log n) \tau^{-\frac{\kappa}{\theta}}(\log n).$$

2. Suppose that $\kappa = \theta \left(\frac{1}{q} - \frac{1}{p} \right)_+$.

- (a) Let $p \geq q$ and $\alpha_0 := \alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right) > 0$ for $\kappa_w > \frac{\theta}{q}$, $\alpha_0 := \alpha - 1 - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right) > 0$ for $\kappa_w = \frac{\theta}{q}$. Then

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 (\log n)^{-\alpha_0} \rho(\log n) \tau^{-\frac{1}{q} + \frac{1}{p}}(\log n).$$

- (b) Let $p < q$, $\alpha_0 := \alpha > 0$ for $\kappa_w > \frac{\theta}{q}$ and $\alpha_0 := \alpha - \frac{1}{q} > 0$ for $\kappa_w = \frac{\theta}{q}$. In addition, suppose that $\alpha_0 \neq \frac{1}{p} - \frac{1}{q}$. Then

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\alpha_0 - \frac{1}{q} + \frac{1}{p}} \rho(\log n)$$

$$\text{if } \alpha_0 > \frac{1}{p} - \frac{1}{q},$$

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 n^{-\alpha_0} \rho(n)$$

$$\text{if } \alpha_0 < \frac{1}{p} - \frac{1}{q}.$$

If $\theta = 0$, $\kappa = 0$, then we suppose that $\rho_u(t) = |\log(t+2)|^{-\lambda_u}$, $\rho_w = |\log(t+2)|^{-\lambda_w}$, $\tau(t) = |\log(t+2)|^\nu$. Denote $\lambda = \lambda_u + \lambda_w$. If $\kappa_w > 0$, $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ > 0$, then the bounds for $C(j)$ from the sharp two-sided estimate of $\mathfrak{S}_{\mathcal{A}_{n_{j,i}}, u, w}^{p, q}$ are already obtained. If $\kappa_w > 0$, $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ = 0$, $\lambda > (1 - \nu) \left(\frac{1}{q} - \frac{1}{p} \right)_+$, then $C(j)$ for $j \geq 2$ is defined as follows. If $p \leq q$, then $C(j) = [\log(m_*j)]^{-\lambda}$; if $p > q$, then

$$C(j) = (m_*j)^{\gamma(\frac{1}{q} - \frac{1}{p})} [\log(m_*j)]^{-\lambda + \frac{1}{q} - \frac{1}{p}}.$$

It follows from estimates in [40], [46].

Theorem 7. Suppose that $\theta = 0$, $\kappa = 0$, $\kappa_w > 0$.

1. Let $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ > 0$. Then

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 n^{-\frac{\alpha}{1-\gamma} + \frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda - \frac{\alpha\nu}{1-\gamma}}.$$

2. Let $\alpha - (1 - \gamma) \left(\frac{1}{q} - \frac{1}{p} \right)_+ = 0$, $\lambda > (1 - \nu) \left(\frac{1}{q} - \frac{1}{p} \right)_+$.

(a) Suppose that $p \geq q$. Then

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 (\log n)^{-\lambda + (1-\nu)\left(\frac{1}{q} - \frac{1}{p}\right)}.$$

(b) Let $p < q$, $\lambda \neq \frac{1}{p} - \frac{1}{q}$. Then

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 n^{\frac{1}{q} - \frac{1}{p}} (\log n)^{-\lambda + \frac{1}{p} - \frac{1}{q}}$$

for $\lambda > \frac{1}{p} - \frac{1}{q}$,

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \rightarrow l_q(\mathcal{A})) \asymp_3 n^{-\lambda}$$

for $\lambda < \frac{1}{p} - \frac{1}{q}$.

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